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634. ADDENDA TO THE MONOGRAPH „ANALYTIC INEQUALITIES" II*

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PART II: ON SOME CONVEX SEQUENCES CONNECTED WITH N. OZEKI'S RESULTS

This is a research-expository paper with certain additions and contributions to the treated matter. In the paper some priorities are also established and the results of OZEKI are critically exposed.

0. Introduction. N. OZEKI published seven papers ([1]–[7]) on matters related to convex sequences and their inequalities. Excepting those numbered [1] and [2], which were reviewed rather incompletely[1], the others were not even reported in the *Mathematical Reviews*. What is more, to the best of our knowledge, both the *Zentralblatt für Mathematik* and the *Referativnyi Žurnal Matematika* seldom report on what appears in the Japanese periodical called *Journal of College of Arts and Sciences*, Chiba University (which is where all OZEKI's papers have been printed). We required a lot of time and considerable effort, trying to get hold of OZEKI's papers. Unfortunately, three of the papers ([1], [3] and [4]) were printed in Japanese, which meant extra difficulty for us to overcome.

In OZEKI's papers there are some attractive new results but they contain several actual mistakes which have had to be resolved first, before we could arrive at the theorems he has established. OZEKI provides an extremely small number of references, and it appears that his ideas seldom stem from the achievements described in mathematical literature. OZEKI arrived yet again at some of the known results, creating interesting and simple methods for proving them. In some instances he arrived at conclusions lees plausible than those already established. Reading OZEKI's papers involves a certain amount of strain on account of the misprints in which they abound.

It is almost an axiom that results which are not well exposed in at least one review journal are almost lost for mathematics. This is even more so if a paper is not annoted at all in those journals and that is not a rare event. The following exposition attempts to review OZEKI's results critically and also to fill in the gaps, and at the same time provides extracts from other mathematicians' papers which delved in matters related to convex sequences. It is thus made possible for those from the mathematical circuit to acquaint themselves with OZEKI's achievements.

1. Some results for convex sequences

In OZEKI's papers one of the basic concepts is the concept of a convex sequence. That is why, at the beginning of this exposition, we shall give several definitions which will be used troughout this paper.

Let (a_n) $(n=1, 2, \ldots)$ be a real sequence. The k-th order difference of sequence (a_n) is defined by

$$\Delta^0 a_n = a_n, \quad \Delta^k a_n = \Delta^{k-1} a_{n+1} - \Delta^{k-1} a_n \qquad (k=1, 2, \ldots).$$

Instead of Δ^1 we shall write Δ.

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* Presented in 1977 by P. R. BEESACK, P. S. BULLEN and A. LUPAS.

[1] See: *Mathematical Reviews* **34** (1967), review 6009 by S. IZUMI and **39** (1970), review by U. C. GUHA.

The following definition introduce the notion of a convex sequence of order $k\,(k=0,\ 1,\ \ldots)$.

Definition 1. *A sequence (a_n) is said to be convex of ordes k if $\Delta^k a_n \geqq 0$ for all $n \in \mathrm{N}$. Particularly, a convex sequence of order $k=2$ is said to be convex.*

On the basis of def. 1, the following definition can be introduced.

Definition 2. *A positive sequence (a_n) is said to be logarithmically convex of order $k\,(k=0,\ 1,\ \ldots)$ if the sequence $(\log a_n)$ is convex of order k.*

One of the first results of OZEKI which we encountered, relevant to convex sequences, is the following theorem proved in [1] (see also [8], p. 202).

Theorem 1. *Let (a_n) be a real sequence and let the sequences (A_n) and (B_n) be defined by*

$$A_n = \frac{1}{n} \sum_{k=1}^{n} a_k, \quad B_n = \Delta^2 A_n \qquad (n=1,\ 2,\ \ldots). \tag{1}$$

Then, if the sequence (a_n) is convex, the following holds

$$B_n \geqq \frac{n-1}{n+2}\, B_{n-1} \qquad (n=2,\ 3,\ \ldots), \tag{i}$$

(ii) *sequence (A_n) is convex, i.e. $\Delta^2 A_n \geqq 0\ (n=1,\ 2,\ \ldots)$ is valid*

In the above mentioned paper [1] OZEKI gave a proof of this theorem, by elementary methods. A more general theorem is proved below (see theorem 4) and for this reason the proof of theorem 1 is omitted here. Let us mention only that the proof of statement (ii) follows directly by repeated application of the inequality quoted under (i) and the fact that $B_2 = \frac{1}{3}\,\Delta^2 a_1$.

An assertion similar to that of (ii), theorem 1, is valid for logarithmically convex sequences, i.e. OZEKI in [1] proved the following theorem:

Theorem 2. *If the sequence (a_n) is logarithmically convex, then the sequence (A_n), defined by* (1), *is also logarithmically convex, i.e. the implication*

$$(a_{n+1})^2 \leqq a_n a_{n+2} \ \Rightarrow\ (A_{n+1})^2 \leqq A_n A_{n+2} \qquad (n=1,\ 2,\ \ldots)$$

is valid.

Later on, we shall give a more general theorem together with its proof (see theorem 7), so that the proof of theorem 2 is omitted.

As stated by OZEKI himself in [1] these two theorems represent answers to problems set by RYO HIROKAWA.

Summarizing these results in [1] OZEKI quotes the following list of implications:

$$\begin{array}{ccc} (a_{n+1})^2 \leqq a_n a_{n+2} & \Rightarrow & a_{n+2} - 2\,a_{n+1} + a_n \geqq 0, \\ \Downarrow & & \Downarrow \\ (A_{n+1})^2 \leqq A_n A_{n+2} & \Rightarrow & A_{n+2} - 2\,A_{n+1} + A_n \geqq 0, \end{array}$$

where the quoted inequalities hold for evevry $n \in \mathrm{N}$ and the sequence (A_n) is defined by (1). Let us mention that implication $\Delta^2 \log a_n \geqq 0 \Rightarrow \Delta^2 a_n \geqq 0$ was already proved in 1928 by MONTEL [9], where, among other results, a necessary and sufficient condition for the logarithmic convexity of a real sequence was given (see also [8], p. 19, as well as the literature related to it).

Naturally, a question arises whether assertion (ii) of theorem 1 could be extended to to the class of convex sequences of order $k \geqq 3$. In paper [6] an answer to that question is provided in the form of the following theorem (see theorem 3, p. 3, in [6]).

Theorem 3. *Let (a_n) be a positive sequence. Then from the k-th order convexity of sequence (a_n), follows the k-th order convexity of the sequence (A_n), where A_n is defined by* (1).

Prior to proceeding to the proof of this theorem, we have to make several remarks. In the proof quoted in [6] the assumption of the positivity of the sequence (a_n) was not used, which means that theorem 3 holds for arbitrary real sequences. On the other hand, while the basic idea of the proof is carried out with a series of errors. Some of these errors are removed when proving theorem 3 in [7], though we must state that this is not explicitly stressed.

Retaining OZEKI's basic idea, we shall give a shorter version of the proof of theorem 3.

Proof. It is easily verified that equality

$$(n+k)\,\Delta^k A_n = (n-1)\,\Delta^k A_{n-1} + \Delta^k a_n \qquad (n=2,\ 3,\ \ldots) \tag{2}$$

holds (OZEKI's correct proof of (2), which is somewhat longer, is given in lemma 2, on page 3 of [7]). By a straight forward calculation we find

$$\Delta^k A_1 = \frac{1}{k+1}\,\Delta^k a_1. \tag{3}$$

By a successive applications of formula (2) for $n=2, 3, \ldots$ and by (3), we find that inequality

$$\binom{n+k}{k}\,\Delta^k A_n \geqq \frac{1}{n}\,\Delta^k a_1 \geqq 0 \qquad (n=1,\ 2,\ \ldots)$$

is valid, which completes the proof of theorem 3. □

The proof of equality (2) given in [6] is not correct. On the other hand, the same equality is correctly proved in [7], but this second proof of OZEKI is complicated. By applying the well known formula (see, for example, [10], pp. 7—8)

$$\Delta^k a_n b_n = \sum_{i=0}^{k} \binom{k}{i} \Delta^i a_n \Delta^{k-i} b_{n+i}$$

to equality $nA_n = \sum_{k=1}^{n} a_k$ (which directly follows from (1)), (2) follows directly.

Starting from (ii) in theorem 1, VASIĆ, KEČKIĆ, LACKOVIĆ and MITROVIĆ [11] proved that the statement of theorem 3 holds without assumption on the positivity of sequence (a_n). The proof given in [11] is considerably shorter then

OZEKI's proof given in [6]. OZEKI could not know the result obtained in [11], because this paper, communicated in 1970, was published in 1972.

The results obtained in theorems 1, 2 and 3 are generalized in another direction, too, in papers [2], [11] and [12].

Let us consider a triangular matrix of real numbers $(p_{n,j})$ (where $j=0, 1, \dots, n$; $n=0, 1, \dots$). Let us define the sequence (σ_n), for a given sequence (a_n) by

$$\sigma_n = \sum_{j=0}^{n} p_{n,n-j}\, a_j. \tag{4}$$

In paper [2] OZEKI obtained the conditions on a triangular matrix $(p_{n,j})$, implying that for each convex sequence (a_n) the sequence (σ_n) defined by (4) is also convex. This result of OZEKI reads:

Theorem 4. *A necessary and sufficient condition that the implication*

$$\Delta^2 a_n \geqq 0 \;\Rightarrow\; \Delta^2 \sigma_n \geqq 0$$

is valid, for every sequence (a_n), where the sequence (σ_n) is given by (4), is that the following conditions, for every n,

$$\alpha_{n-1,n-1} - 2\alpha_{n,n} + \alpha_{n+1,n+1} = 0, \tag{5}$$

$$\beta_{n-1,n-2} - 2\beta_{n,n-1} + \beta_{n+1,n} = 0, \tag{6}$$

$$\beta_{n-1,n-k-1} - 2\beta_{n,n-k} + \beta_{n+1,n-k+1} \geqq 0 \qquad (k=2, \dots, n-1), \tag{7}$$

$$\beta_{n+1,1} - 2\beta_{n,0} \geqq 0, \tag{8}$$

$$\beta_{n+1,0} \geqq 0 \tag{9}$$

hold, where

$$\alpha_{n,k} = \sum_{j=0}^{k} p_{n,j}, \tag{10}$$

$$\beta_{n,k} = \sum_{j=0}^{k} \alpha_{n,j}. \tag{11}$$

In paper [2] OZEKI gave only the proof of the sufficiency of the conditions (5) — (9). However, we shall give a proof that the conditions (5) — (9) are also necessary.

Proof. (i) Conditions (5) — (9) are sufficient. Let us define the sequence (e_k) $(k=0, 1, \dots)$ in the following manner

$$e_0 = a_0, \quad e_1 = \Delta a_0, \quad e_{k+1} = \Delta^2 a_{k-1} \qquad (k=1, 2, \dots). \tag{12}$$

On the basis of the assumption that the sequence (a_n) is convex we find that $e_k \geqq 0$ $(k=2, 3, \dots)$, while e_0 and e_1 can be of an arbitrary sign. On the basis of (12) we have

$$a_n = e_0 + n e_1 + (n-1) e_2 + (n-2) e_3 + \cdots + e_n \qquad (n=0, 1, \dots). \tag{13}$$

Applying (4) and (13) we get $\sigma_n=\sum_{j=0}^{n}\alpha_{n,n-j}e_j$, which upon evident transformations gives

$$(14)\quad \Delta^2\sigma_{n-1,n-1}=(\alpha_{n-1,n-1}-2\alpha_{n,n}+\alpha_{n+1,n+1})e_0 + (\beta_{n-1,n-2}-2\beta_{n,n-1}+\beta_{n+1,n})e_1 + \sum_{k=2}^{n}(\beta_{n-1,n-k-1}-2\beta_{n,n-k}+\beta_{n+1,n-k+1})e_k + (\beta_{n+1,1}-2\beta_{n,0})e_n+\beta_{n+1,0}e_{n+1}.$$

From the above equality and conditions (5) — (9) it follows that $\Delta^2\sigma_{n-1}\geqq 0$. Which proves that the (5) — (9) are sufficient.

(ii) Conditions (5) — (9) are necessary. Let us choose, first, the sequence (a_n) defined by $a_n=1$ $(n=0, 1,\ldots)$. This sequence is convex and $e_0=1$, $e_k=0$ $(k=1, 2,\ldots)$. Since, by assumption, for any convex sequence (a_n) the sequence (σ_n) defined by (4) is convex, too, we find on the basis of (14) that

$$(15)\quad \Delta^2\sigma_{n-1}=\alpha_{n-1,n-1}-2\alpha_{n,n}+\alpha_{n+1,n+1}\geqq 0.$$

Analogously, if we select $a_n=-1$ $(n=0, 1,\ldots)$ (this is also a convex sequence), we get $e_0=-1$, $e_k=0$ $(k=1, 2,\ldots)$ so that from the assumed implication on the convexity of the sequences (a_n) and (σ_n), on the basis of (14) we have

$$(16)\quad \Delta^2\sigma_{n-1}=-(\alpha_{n-1,n-1}-2\alpha_{n,n}+\alpha_{n+1,n+1})\geqq 0.$$

On the grounds of (15) and (16) we infer that (5) is valid. From this it follows that (14) becomes

$$(17)\quad \Delta^2\sigma_{n-1}=(\beta_{n-1,n-2}-2\beta_{n,n-1}+\beta_{n+1,n})e_1 + \sum_{k=2}^{n}(\beta_{n-1,n-k-1}+2\beta_{n,n-k}+\beta_{n+1,n-k})e_k + (\beta_{n+1,1}-2\beta_{n,0})e_n+\beta_{n+1,0}e_{n+1}.$$

Now let us assume that $a_k=k$ $(k=0, 1,\ldots)$. Since $\Delta^2 a_k\geqq 0$, we have $\Delta^2\sigma_k\geqq 0$ Therefrom, since $e_0=e_k=0$ $(k=2, 3,\ldots)$ and $e_1=1$, we get

$$(18)\quad \Delta^2\sigma_{n-1}=\beta_{n-1,n-2}-2\beta_{n,n-1}+\beta_{n+1,n}\geqq 0.$$

On the other hand, in the case when $a_k=-k$ $(k=0, 1,\ldots)$, since $e_0=e_k=0$ $(k=2, 3,\ldots)$ and $e_1=-1$, we find that

$$(19)\quad \Delta^2\sigma_{n-1}=-(\beta_{n-1,n-2}-2\beta_{n,n-1}+\beta_{n+1,n})\geqq 0.$$

From inequalities (18) and (19) we see that the condition (6) holds. In the other words, equalities (14), i.e. (17), take the form

$$(20)\quad \Delta^2\sigma_{n-1}=\sum_{k=2}^{n-1}(\beta_{n-1,n-k-1}-2\beta_{n,n-k}+\beta_{n+1,n-k+1})e_k + (\beta_{n+1,1}-2\beta_{n,0})e_n+\beta_{n+1,0}e_{n+1}.$$

Now let us choose sequences (a_k^j) $(j, k=0, 1, \ldots)$ so that

$$a_k^j=0 \quad (0\leqq k\leqq j-1), \qquad a_k^j=k-j+1 \quad (k\geqq j).$$

For the j-th sequence we have, on the basis of (12),

$$e_k^j=0 \quad (k\neq j), \quad e_k^k=1.$$

Using the sequence (a_k^j) $(j=2, \ldots, n+1)$ we find on the basis of (20) that

$$\text{(21)} \qquad \Delta^2 \sigma_{n-1}=\begin{cases} \beta_{n-1,\, n-j-1}-2\,\beta_{n,\, n-j}+\beta_{n+1,\, n-j+1} & (2\leqq j\leqq n-1), \\ \beta_{n+1,\, 1}-2\,\beta_{n,\, 0} & (j=n), \\ \beta_{n+1,\, 0} & (j=n+1). \end{cases}$$

Since the sequences (a_n^j) $(2\leqq j\leqq n+1)$ are convex, then $\Delta^2 \sigma_{n-1}\geqq 0$, so that by (21) we get, in turn, conditions (7)—(9). This completes the proof of theorem 4. □

From OZEKI's paper [2] it cannot be seen why the proof that conditions (5)—(9) are necessary, is omitted (part (ii) of the proof of theorem 4) because the same is not at all simpler than the first part of the proof of theorem 4. Otherwise this idea, presented in part (ii) is taken from [11] and [12]. In [11] a transformation of a sequence (a_n) of the form

$$\text{(22)} \qquad A_n=\frac{\sum\limits_{k=0}^{n} p_k a_k}{\sum\limits_{k=0}^{n} p_k} \qquad (n=0, 1, \ldots),$$

is considered, where $p_k>0$ $(k=0, 1, \ldots)$. The following theorem was proved for the sequences (a_n) and (A_n).

Theorem 5. *A necessary and sufficient condition that for every sequence (a_n) the implication*

$$\Delta^2 a_n\geqq 0 \ \Rightarrow\ \Delta^2 A_n\geqq 0 \qquad (n=0, 1, \ldots)$$

is valid, where the sequence (A_n) is given by (22), is that the sequence (p_n) (of positive weights p_n) is of the form

$$\text{(23)} \qquad p_n=\frac{\prod\limits_{i=1}^{n-1}(p_1+(i-1)p_0)}{p_0^{n-1}(n-1)!} \qquad (n=2, 3, \ldots),$$

where p_0 and p_1 are arbitrary positive numbers.

Let us observe that the sequence (A_n), given by (22), is of the form (4), if the triangular matrix $(p_{n,j})$ is chosen in such a way that

$$p_{n,k}=\frac{p_{n-k}}{\sum\limits_{j=0}^{n} p_j} \qquad (k=0, 1, \ldots, n; \quad n=0, 1, \ldots).$$

Theorem 5 is a special case of theorem 4, namely each sequence (p_n) of the form (23) satisfies conditions (5)—(9), where the sequences $(\alpha_{n,j})$ and $(\beta_{n,j})$ are defined by (10) and (11). The sole advantage of theorem 5 with respect to theorem 4 is that the weights (p_n) in theorem 5 are explicitly obtained and conditions (5)—(9) are quite complicated to verify. It is certainly made possible by an additional condition on matrix $(p_{n,j})$, which reads $\sum_{k=0}^{n} p_{n,k}=1$, where $p_{n,k}>0$.

LACKOVIĆ and SIMIĆ [12] made the following generalization of theorem 5.

Theorem 6. *A necessary and sufficient condition that the implication*

$$\Delta^r a_n \geqq 0 \Rightarrow \Delta^r A_n \geqq 0 \qquad (n=0, 1, \ldots)$$

is valid for every sequence (a_n) and sequence (A_n) given by (22), is that the sequence (p_n), for $n=r, r+1, \ldots,$ is of the form

$$\text{(24)} \qquad p_n=\frac{(r-1)!\,p_{r-1}}{n!\,(p_0+\cdots+p_{r-2})^{n-r+1}}\prod_{k=r-2}^{n-2}((k+1)(p_0+\cdots+p_{r-2})+(r-1)p_{r-1}),$$

where $p_0, \ldots, p_{r-1}$ are arbitrary positive numbers and $r\geqq 2$ is a fixed natural number.

As stated by OZEKI himself in theorem 4 from [2], the following assertions are immediate:

(a) If sequence (a_n) is convex, then the sequence

$$\sigma_n=\frac{a_0+\cdots+a_n}{n+1}$$

is also convex.

(b) If the sequence (a_n) is convex, then the sequence of HÖLDER's means is convex too. HÖLDER's means are defined by

$$H_n^1=\frac{a_0+\cdots+a_n}{n+1}, \qquad H_n^{k+1}=\frac{H_0^k+\cdots+H_n^k}{n+1} \qquad (k=1, 2, \ldots).$$

(c) If the sequence (a_n) is convex then the same property is possessed by the sequence (σ_n) defined by

$$\sigma_n=\frac{a_0+\binom{n}{1}a_1+\cdots+\binom{n}{n}a_n}{2^n}.$$

All the quoted particular cases are also consequences of theorem 5. In paper [2] OZEKI gave some simple assertions for CESÀRO means (σ_n^α) of the sequence (a_n). In [11] the connection of these theorema with the sequences of bounded variation is given.

In [2] OZEKI presented (see theorem 2 from [2]) a theorem for logarithmically convex sequences analogous to theorem 4. Prior to proceeding to the statement and the proof of this theorem we shall introduce some notations:

Let the sequences (p_n) and (P_n) $(n=0, 1, \ldots)$ be strictly positive and let

$$q_k=\frac{p_{k-1}p_{k+1}}{p_k^2}, \qquad Q_k=\frac{P_{k-1}P_{k+1}}{P_k^2} \qquad (k=1, 2, \ldots).$$

For a given sequence (a_n) let the sequence (σ_n) be defined by

$$\sigma_n=\frac{p_0 a_0+\cdots+p_n a_n}{P_n} \qquad (n=0, 1, \ldots). \tag{25}$$

Theorem 7. *Let us assume that the sequences* (p_n) *and* (P_n) *satisfy the conditions*

$$Q_0=0, \tag{26}$$

$$2Q_n>1>Q_n \qquad (n=1, 2, \ldots), \tag{27}$$

$$q_n \geqq Q_n \qquad (n=1, 2, \ldots), \tag{28}$$

$$(1-Q_{n-1}Q_n)^2 \leqq 4q_n(1-Q_{n-1})(1-Q_n) \qquad (n=1, 2, \ldots). \tag{29}$$

Then, if a positive sequence (a_n) *is logarithmically convex, the sequence* (σ_n), σ_n *being defined by* (25), *is also positive and logarithmically convex. In other words, the implication*

$$a_n^2 \leqq a_{n+1}a_{n-1} \Rightarrow \sigma_n^2 \leqq \sigma_{n+1}\sigma_{n-1} \qquad (n=1, 2, \ldots)$$

is valid.

Proof. Let

$$t_0 t_1 \cdots t_n = p_0 a_0+\cdots+p_n a_n \qquad (n=0, 1, \ldots). \tag{30}$$

The sequence (t_n) is well defined by (30) and on the basis of that relation we immediately have $t_n>0$. From the same relation it follows that

$$p_n a_n = t_0 t_1 \cdots t_{n-1}(t_n-1),$$

so that with respect to the inequality $p_n a_n>0$ we get $t_n>1$. Since we have assumed that $a_{n-1}a_{n+1}\geqq a_n^2$, we have

$$t_{n+1}\geqq 1+\frac{q_n t_{n-1}(t_n-1)^2}{t_n(t_{n-1}-1)}. \tag{31}$$

Using (25), the definition of the sequence (t_n) and relation (31) we have

$$\begin{aligned}
\sigma_{n-1}\sigma_{n+1}-\sigma_n^2 &= (t_0 t_1\cdots t_{n-1})^2 t_n\left(\frac{t_{n+1}}{P_{n-1}P_{n+1}}-\frac{t_n}{P_n^2}\right)\\
&= (t_0 t_1\cdots t_{n-1})^2\frac{t_n}{P_{n-1}P_{n+1}}(t_{n+1}-t_n Q_n)\\
&= C(t_{n+1}-t_n Q_n)\\
&\geqq C\left(\frac{q_n t_{n-1}(t_n-1)^2}{t_n(t_{n-1}-1)}+1-t_n Q_n\right)\\
&= C_1((q_n t_{n-1}-Q_n t_{n-1}+Q_n)t_n^2+(t_{n-1}-1-2q_n t_{n-1})t_n+q_n t_{n-1}),
\end{aligned}$$

where $C \geqq 0$ and $C_1 \geqq 0$. Let us mention that we have introduced the notation

$$C_1 = \frac{C}{t_n(t_{n-1}-1)} > 0 \qquad (n>1)$$

and furthermore let

$$(32) \qquad f(t_n) = (q_n t_{n-1} - Q_n t_{n-1} + Q_n) t_n^2 + (t_{n-1} - 1 - 2 q_n t_{n-1}) t_n + q_n t_{n-1}.$$

On the basis of (28) we see that $q_n t_{n-1} - Q_n t_{n-1} + Q_n > 0$. The discriminant of $f(t), f$ being given by (32), has the form

$$D = (t_{n-1} - 1)\big(t_{n-1}(1 - 4 q_n (1 - Q_n)) - 1\big).$$

If $D \leqq 0$ then it immediately follows that $\sigma_{n-1}\sigma_{n+1} - \sigma_n^2 \geqq 0$. Let us consider the case $D > 0$. Since $t_{n-1} > 1$ we see that $t_{n-1}(1 - 4 q_n (1 - Q_n)) > 1$ and from $t_{n-1} > 1$ we find that $1 - 4 q_n (1 - Q_n) > 0$, i. e. the condition $D > 0$ implies

$$(33) \qquad t_{n-1} > \frac{1}{1 - 4 q_n (1 - Q_n)}.$$

The proof that in this case also $\sigma_{n-1}\sigma_{n+1} - \sigma_n^2 \geqq 0$ holds, will be continued by induction. Let $n = 1$. We shall prove that $\sigma_0 \sigma_2 - \sigma_1^2 \geqq 0$. It is directly verified that

$$\sigma_0 \sigma_2 - \sigma_1^2 = \frac{(p_0 a_0)^2}{P_0 P_2}\left(-Q_1 t^2 + (1 - 2 Q_1) t + 1 - Q_1 + \frac{p_2 a_2}{p_0 a_0}\right)$$

is valid, where the notation $t = \frac{p_1 a_1}{p_0 a_0} > 0$ was introduced. Since $C = \frac{(p_0 a_0)^2}{P_0 P_2} > 0$, then we have

$$\sigma_0 \sigma_2 - \sigma_1^2 = C\left((q_1 - Q_1) t^2 + (1 - 2 Q_1) t + (1 - Q_1) - q_1 t^2 + \frac{p_2 a_2}{p_0 a_0}\right)$$

$$= C\left((q_1 - Q_1) t^2 + (1 - 2 Q_1) t + 1 - Q_1 + \frac{p_2 (a_0 a_2 - a_1^2)}{p_0 a_0^2}\right)$$

$$\geqq C((q_1 - Q_1) t^2 + (1 - 2 Q_1) t + 1 - Q_1) = C f_1(t)$$

(let us mention that this last relation is not proved correctly by OZEKI in [2]). The discriminant of the polynomial $f_1(t)$ is

$$D_0 = (1 - 2 Q_1)^2 - 4 (1 - Q_1)(q_1 - Q_1) = 1 - 4 q_1 (1 - Q_1).$$

Using conditions (26) and (29) for $n = 1$, we find that $D_0 \leqq 0$, i. e. $f_1(t) \geqq 0$. Thereby it is proved that $\sigma_0 \sigma_2 - \sigma_1^2 \geqq 0$.

Let us further suppose that

$$(34) \qquad \sigma_{n-2}\sigma_n - \sigma_{n-1}^2 = C(t_n - Q_{n-1} t_{n-1}) \geqq 0,$$

where the positive quantity C was defined earlier. It is immediately verified that

$$f(Q_{n-1} t_{n-1}) = t_{n-1}\big((q_n - Q_n) Q_{n-1}^2 t_{n-1}^2 + (Q_{n-1} Q_n + 1 - 2 q_n) Q_{n-1} t_{n-1} + (q_n - Q_{n-1})\big)$$

$$= t_{n-1} F(t_{n-1})$$

holds. The discriminant of the quadratic polynomial $F(t)$ is

$$D_1=(1-Q_{n-1}Q_n)^2-4q_n(1-Q_n)(1-Q_{n-1})$$

and on the basis of the assumption (29) we have $D_1 \leqq 0$. In other words, we find that

$$f(Q_{n-1}t_{n-1})\geqq 0. \tag{35}$$

We have assumed that $D>0$, which means that equation $f(t)=0$, where f is defined by (32), has two distinct real roots α and β. Further we have

$$Q_{n-1}t_{n-1}-\frac{\alpha+\beta}{2}=\frac{1}{2((q_n-Q_n)t_{n-1}+Q_n)}\big(2Q_{n-1}(q_n-Q_n)t_{n-1}^2$$
$$+(2Q_nQ_{n-1}+1-2q_n)t_{n-1}-1\big)=C_2f_2(t_{n-1}),$$

where C_2 is positive. For the so defined quadratic trinomial $f_2(t)$ we have

$$f_2(0)=-1. \tag{36}$$

By a direct calculation we find that

$$f_2\Big(\frac{1}{1-4q_n(1-Q_n)}\Big)=C_3(2Q_n-1)(Q_{n-1}(2Q_n-1)-1+4q_n(1-Q_n)), \tag{37}$$

where, on the basis of (27), $2Q_n>1$ and the quantity C_3 is also positive. On the basis of (29) we have

$$4q_n(1-Q_n)\geqq\frac{(1-Q_{n-1}Q_n)^2}{1-Q_{n-1}}$$

so that from (37) we get

$$f_2\Big(\frac{1}{1-4q_n(1-Q_n)}\Big)\geqq C_4(Q_{n-1})^2(1-Q_n)^2\geqq 0, \tag{38}$$

where C_4 is positive. On the basis of (33), (35), (36) and (38) we get

$$Q_{n-1}t_{n-1}-\frac{\alpha+\beta}{2}=C_2f_2(t_{n-1})>0. \tag{39}$$

From (34) it follows that $t_n \geqq t_{n-1}Q_{n-1}$. Hence on the basis of (35) and (39) we get

$$\sigma_{n-1}\sigma_{n+1}-\sigma_n^2=C_1f(t_n)\geqq 0,$$

which completes the induction proof of OZEKI's theorem 7. □

Let us note that theorem 2 is a special case of theorem 7. Namely, if in theorem 7 we take $P_n=1$ and $p_0=p_1=\cdots=p_n=1$ for all $n=0, 1, \ldots$, then assumptions (26) — (29) are satisfied and from the logarithmic convexity of sequence (a_n) the logarithmic convexity of sequence (A_n) follows, A_n being given by (1).

If the sequence (a_n) $(n=0, 1, \ldots)$ is convex, where $a_0=0$, it is simple to prove that the sequence $\frac{1}{n}a_n$ $(n=1, 2, \ldots)$ is nondecreasing. Namely, from

$a_0=0$ and $\Delta^2 a_0 \geqq 0$ we find that $\frac{1}{2} a_2 \geqq a_1$. On the other hand, since $\Delta^2 a_{n-1} \geqq 0$, we get

$$\frac{a_{n+1}}{n+1}-\frac{a_n}{n} \geqq \frac{n-1}{n+1}\left(\frac{a_n}{n}-\frac{a_{n-1}}{n-1}\right)$$

so that the above assertion follows by induction. In a similar way if a positive sequence (a_n) $(n=0, 1, \ldots)$ $(a_0=1)$ is logarithmically convex, then the sequence $\sqrt[n]{a_n}$ $(n=1, 2, \ldots)$ is nondecreasing. This shows, for example, that convexity of (a_n) is a stronger condition than monotony of the sequence $\left(\frac{1}{n} a_n\right)$ is. In references to the above we can raise a question whether this weaker condition entails the analogous behaviour of arithmetic means defined by (1). The answer to this problem is contained in the following two theorems from [3].

Theorem 8. *Let the sequence (b_n) be nondecreasing and let the sequence (B_n) be defined by*

(40) $$B_n=\frac{1}{n} \sum_{k=1}^{n} k b_k .$$

Then

(i) $(n+1) b_n \geqq 2 B_n$ *and* (ii) $\frac{B_{n+1}}{n+2} \geqq \frac{B_n}{n+1}$.

Proof. (i) On the basis of (40) we have

$$(n+1) B_{n+1}=n B_n+(n+1) b_{n+1},$$

i.e.

(41) $$(n+2) b_{n+1}-2 B_{n+1}=n\left(b_{n+1}-\frac{2 B_n}{n+1}\right) \geqq \frac{n}{n+1}((n+1) b_n-2 B_n).$$

Since $3 b_2-2 B_2=b_2-b_1 \geqq 0$ assertion (i) follows from (41) by induction.
(ii) Since

$$(n+1) B_{n+1}-(n+2) B_n=(n+1) b_{n+1}-2 B_n \geqq(n+1) b_n-2 B_n,$$

on the basis of (i), (ii) follows.

Theorem 9. *Let the sequence (b_n) be nondecreasing and let*

$$B_n=\frac{1}{n} \sum_{k=1}^{n} b_k^{k} \quad (b_0=1).$$

Then $\sqrt[n-1]{B_n} \leqq \sqrt[n]{B_{n+1}}$ $(n=2, 3, \ldots)$, *and particularly* $B_2 \geqq \frac{1}{2} B_1$ *is true.*

Theorem 9 is proved (by induction) in a similar way to theorem 8. Let us note that $b_n \geqq 1$ stems from the assumptions of this theorem.

In the particular case, if we take that $b_1=\cdots=b_n=x>0$, by theorem 9 we see that the inequality

$$\left(\frac{1+x+\cdots+x^n}{n}\right)^{\frac{1}{n-1}} \leqq\left(\frac{1+x+\cdots+x^{n+1}}{n+1}\right)^{\frac{1}{n}}$$

holds.

2. Some results concerning the convolution of sequences

Let two sequences (a_n) and (b_n) be given and let the sequence $(\bar{c}_n)$ be defined by

$$\bar{c}_n = \sum_{k=0}^{n} a_k b_{n-k}. \tag{42}$$

The sequence $(\bar{c}_n)$ is called the convolution of the sequences (a_n) and (b_n). Formula (42) gives the coefficients of the development

$$\sum_{k=0}^{+\infty} \bar{c}_k x^k = x^k = \left(\sum_{k=0}^{+\infty} a_k x^k\right)\left(\sum_{k=0}^{+\infty} b_k x^k\right).$$

In many papers (see [14] — [19], as well as the references given in them) the behaviour of sequence (42) was dealt with under different assumptions for sequence (a_n) and (b_n). Some of the quoted papers treat sequences defined by means of $(\bar{c}_n)$ given by (42).

Ozeki's papers contain also a certain number of theorems relevant to such combination of sequences (a_n) and (b_n). So, for example, in paper [2] the following assertion is proved.

Theorem 10. *If a positive sequence $(a_n)\,(n=0, 1, \ldots)$ is logarithmically convex, then the sequence $(\sigma_n)\,(n=0, 1, \ldots)$, where σ_n is defined by*

$$\sigma_n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} a_k, \tag{43}$$

is also logarithmically convex.

Together with sequece (43) we can consider the sequence (S_n), where

$$S_n = \sum_{k=0}^{n} \binom{n}{k} a_k \qquad (n=0, 1, \ldots). \tag{44}$$

It is clear that the sequence (43) is logaritmically convex if and only if sequence (44) has the same property. Ozeki's proof, given [2], is based on just such an idea.

However, an even more general result was already known in 1949. Namely, Davenport and Pólya in [16] proved the following

Theorem 11. *Let the sequence (w_n) be defined by*

$$w_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}, \tag{45}$$

where the sequences (a_n) and (b_n) are positive and logarithmically convex. Then the sequence (w_n) is also positive and logarithmically convex.

It is clear that (45), for a positive and logarithmically convex sequence $b_n = 1\ (n=0, 1, \ldots)$, reduces to (44), i.e. Ozeki's theorem 9 is a special case of theorem 11.

In [4] Ozeki considered a sequence related to convolution (42). Namely, if (a_n) and (b_n) are given sequences, let the sequence (c_n) be defined by

$$c_n = \frac{1}{n+1} \sum_{k=0}^{n} a_k b_{n-k} \qquad (n=0, 1, \ldots). \tag{46}$$

In connection with this sequence the following theorems were proved in [4].

Theorem 12. *Let positive sequences* (a_n) *and* (b_n) *be convex and let us suppose that* $a_1 \geqq a_0$ *and* $b_1 \geqq b_0$. *Then the sequence* (c_n) *defined by* (46) *is convex. In other words, if*

$$a_1 \geqq a_0, \ \Delta^2 a_{i-1} \geqq 0, \ \Delta^2 b_{i-1} \geqq 0 \qquad (i = 1, 2, \ldots), \tag{47}$$

then

$$\Delta^2 c_{i-1} \geqq 0 \qquad (i = 1, 2, \ldots).$$

The following theorem concerns logarithmically convex sequences.

Theorem 13. *If positive sequences* (a_n) *and* (b_n) *satisfy the conditions*

$$a_{i-1} a_{i+1} \geqq a_i^2, \quad b_{i-1} b_{i+1} \geqq b_i^2 \quad (i = 1, 2, \ldots)$$

then the sequence (c_n) *defined by* (46) *also satisfies the condition*

$$c_{i-1} c_{i+1} \geqq c_i^2 \qquad (i = 1, 2, \ldots). \tag{48}$$

It is simple to prove that sequences (a_n) and (b_n), satisfying the assumptions of theorem 12, are positive and nondecreasing. This follows from (47) by induction. The proof of theorem 12 given in [4] is very similar to the proof of theorem 7 and we shall not quote it here.

By a direct calculation we can verify that for every sequence (c_n) equality

$$\Delta^2 (n+1) c_n = (n+1) \Delta^2 c_n + 2 \Delta c_{n+1}$$

is valid. This implies that if the sequences (a_n) and (b_n) satisfy the conditions of theorem 12, then not only the sequence (c_n), given by (46), is convex, but even convolution (42) of sequences (a_n) and (b_n) is convex. The related problems were studied in [17].

The proof of theorem 13 given in [4] is also very similar to that of theorem 7 so that this proof will not be quoted here. In [18] it was noticed that if positive sequences (a_n) and (b_n) are logarithmically convex, then their convolution (42) needs not, in general, be logarithmically convex. This is not contrary to Ozeki's theorem 13. The sequence (c_n), defined by (46), can be written in the form $c_n = \frac{1}{n+1} \bar{c}_n$, where $\bar{c}_n$ is given by (42). By this substitution inequality (48) becomes

$$\frac{(i+1)^2}{i(i+2)} \bar{c}_{i-1} \bar{c}_{i+1} \geqq \bar{c}_i \qquad (i = 1, 2, \ldots),$$

which is a weaker condition than that of the logarithmic convexity. On the other hand if positive sequences (a_n) and (b_n) are logarithmically concave, then their convolution $(\bar{c}_n)$ defined by (42) is also logarithmically concave. In a particular case this result was obtained for the first time by Kaluza [14] in 1928. Somewhat later, in 1933, Karamata [15] again arrived at the same result. Later on the same question was dealt with in papers [16] — [19].

From the above mentioned results for the logarithmically concave sequences we get the following assertion for the sequence (c_n) defined by (46): If positive sequences (a_n) and (b_n) are logarithmically concave, then the sequence (c_n) defined by (46) satisfies the inequality

$$\frac{i(i+2)}{(i+1)^2} c_{i-1} c_{i+1} \leqq c_i^2 \qquad (i=1, 2, \ldots).$$

This condition is now weaker then that of the logarithmic concavity of sequence (c_n).

From the foregoing we infer that theorem 13 cannot be compared with the theorem on the logarithmic concavity of the convolution of positive and logarithmically concave sequences.

3. On the coefficients of Taylor's expansions

In this section we shall present several of Ozeki's results related to functions given in the form of power series with coefficients which are convex or logarithmically convex.

The following definition, introducing a relation among real functions, will considerably shorten this exposition.

Definition 3. *Let us consider two formal power series* $\sum_{k=0}^{+\infty} a_k x^k$ *and* $\sum_{k=0}^{+\infty} b_k x^k$. *These series are said to be in relation* $\gg$, *which is denoted by*

$$\sum_{k=0}^{+\infty} a_k x^k \gg \sum_{k=0}^{+\infty} b_k x^k,$$

if the condition $a_k \geqq b_k$ $(k=0, 1, \ldots)$ *is fulfilled.*

In [3] Ozeki proved the following results related to the relation $\gg$.

Theorem 14. *Let us assume that* $f(x) = \sum_{k=0}^{+\infty} p_k x^k$. *If the positive sequence* (p_k) $(k=0, 1, \ldots)$ *is logatithmically convex, then*

$$\frac{f^{(k-1)}(x)}{(k-1)!} \frac{f^{(k+1)}(x)}{(k+1)!} \gg \left(\frac{f^{(k)}(x)}{k!}\right)^2 \qquad (k=1, 2, \ldots), \tag{49}$$

where the derivatives $f^{(m)}$ *are taken in the formal sense.*

The proof of this theorem, as stated in [3], follows immediately from the calculation of the coefficient of x^n, in the difference of the left and the right sides of (49).

Ozeki [6] also studied some properties of the relation $\gg$ for polynomials. Namely, the following very simple assertions were proved in [6]:

(i) Let $P(x)$, $Q(x)$ and $R(x)$ be polynomials with positive coefficients and of equal degree n. Then

a) $P(x) \gg P(x)$,

b) $P(x) \gg Q(x) \wedge Q(x) \gg P(x) \Rightarrow P(x) = Q(x)$,

c) $P(x) \gg Q(x) \wedge Q(x) \gg R(x) \Rightarrow P(x) \gg R(x)$.

(ii) If $P_n(x)$, $Q_n(x)$, $\overline{P}_m(x)$, $\overline{Q}_m(x)$ are polynomials of degree n and m repectively, with positive coefficients p_i, q_i, $\overline{p}_j$, $\overline{q}_j$ $(i=0, 1, \ldots, n;\ j=0, 1, \ldots, m)$ for which

$$P_n(x) \gg Q_n(x) \quad \text{and} \quad \overline{P}_m(x) \gg \overline{Q}_m(x)$$

holds, then

$$P_n(x)\,\overline{P}_m(x) \gg Q_n(x)\,\overline{Q}_m(x).$$

(iii) If $P(x)$ and $Q(x)$ are arbitrary polynomials of equal degree, then from the condition $P(x) \gg Q(x)$ follows the condition

$$\int_0^x P(x)\,dx \gg \int_0^x Q(x)\,dx.$$

The proofs of these assertions are simple.

Using the properties of relation $\gg$, OZEKI [6] proved

Theorem 15. *Let (a_i), (b_i) and (c_i) be sequences of real numbers for which the conditions*

(50) $$\prod_{i=0}^{n} (x+a_{2i+1}) = \sum_{i=0}^{n} \binom{n+1}{i} b_i x^{n+1-i} \qquad (b_0=1),$$

(51) $$\prod_{i=1}^{n} (x+a_{2i}) = \sum_{i=0}^{n} \binom{n}{i} c_i x^{n-i} \qquad (c_0=1)$$

are valid. Then, if $a_i>0$ and $a_i a_{i+2} - a_{i+1}{}^2 \geqq 0$ $(i=1, 2, \ldots)$, we have $b_i \geqq c_i$ $(i=1, \ldots, n)$.

We will give a somewhat shortened version of OZEKI's proof.

Proof. On the basis of the assumptions of theorem 15, we can see that

(52) $$(x+a_j)(x+a_{j+1}) \gg (x+a_{i+1})(x+a_j) \qquad (j>i)$$

is true, which is obtained directly by comparing polynomial coefficients on the left and right sides of (52).

Further, let us define polynomials P_{2i} and P_{2i+1} by

$$P_{2i}(x) = \prod_{j=1}^{i} (x+a_{2j}), \quad P_{2i+1}(x) = \prod_{j=0}^{i} (x+a_{2j+1}).$$

By induction, on the basis of (52), and using the transitivity of the relation $\gg$, we get

(53) $$P_{2n-1}(x) \gg (x+a_n) P_{2n-2}(x) \qquad (n=2, 3, \ldots)$$

From the logarithmic convexity of the sequence (a_n), by a direct calculation, we see that

(54) $$a_n + n a_{2n+1} - (n+1) a_{2n} \geqq 0 \qquad (n=1, 2, \ldots)$$

holds. If $P(x)=P_{2n-1}(x)+n(x+a_{2n+1})P_{2n-2}(x)-(n+1)P_{2n}(x)$, where P_{2i} and P_{2i+1} are the polynomials defined above, on the basis of transitivity of the relation $\gg$ and (54) we get that $P(x)\gg 0$ i.e.

$$P_{2n-1}(x)+n(x+a_{2n+1})P_{2n-2}(x)-(n+1)P_{2n}(x)\gg 0 \tag{55}$$

is true. By induction, on the basis of (53) and (55), it could be shown that, under the assumption of theorem 15,

$$\left(\prod_{i=0}^{n}(x+a_{2i+1})\right)'\gg(n+1)\prod_{i=1}^{n}(x+a_{2i}) \tag{56}$$

holds. Further, using (50), (51) and (56), on the basis of the property (iii) of relation $\gg$, the assertion of theorem 15 follows.

Ozeki [5] also gave some theoremes related to coefficients of functions given in the form of a prower series. Namely in [5] the following result was proved.

Theorem 16. *Let us assume that the sequence* (q_n) $(n=1, 2, ..)$ *is defined by the following formal equality*

$$1+\sum_{k=1}^{+\infty} q_k x^k=\left(1-\sum_{k=1}^{+\infty} p_k x^k\right)^{-1}, \tag{57}$$

where the sequence (p_n) $(n=1, 2, ...)$ *is given. In other words, let*

$$q_1=p_1, \quad q_n=p_n+\sum_{k=1}^{n-1} p_k q_{n-k} \qquad (n=2, 3, ...); \tag{58}$$

then, if the positive sequence (p_n) *is logarithmically convex, we have*

$$q_n>0 \text{ and } q_n q_{n+2}\geqq q_{n+1}^2 \qquad (n=1, 2, ...). \tag{59}$$

The proof of this theorem, given in [5] by Ozeki, is based on the following theorem given in the same paper.

Theorem 17. *Let* $D_n (n=1, 2, ...)$ *denote the determinant of order* n *with entries* a_{ij}, *where*

$$a_{ij}\geqq 0\ (j\geqq i-1),\ a_{ij}=-1\ (j=i-2),\ a_{ij}=0\ (j<i-2)\ (i, j=1, ..., n).$$

Then, if

$$\begin{vmatrix} a_{pi} & a_{pj} \\ a_{qi} & a_{qj} \end{vmatrix}\leqq 0\ (1\leqq p\leqq q\leqq n;\quad 1\leqq i\leqq j\leqq n),$$

then $(-1)^{n+1}D_n\geqq 0$ $(n=1, 2, ...)$.

The proof of this theorem is carried out by induction in [5].

We wish to mention that the positivity of the coefficients q_n, under the quoted assumptions of theorem 16, was proved already in 1933 in [15]. Similar, but much more general results were obtained in [17].

According to the monograph [20], p. 164, Schur proved the following assertion.

If $a_1, \ldots, a_n$ are positive real numbers and if sequencee (q_n) is defined by

$$\frac{1}{\prod_{i=1}^{n}(1-x a_i)} = 1 + \sum_{i=1}^{+\infty} \binom{n+i-1}{i} q_i x^i, \tag{60}$$

then the sequence (q_n) is logarithmically convex.

OZEKI in [5], parallel with the above expansion, considered the expansion

$$\prod_{i=1}^{n}(1-x a_i) = 1 + \sum_{i=1}^{n} (-1)^i \binom{n}{i} p_i x^i \tag{61}$$

and proved the following result:

Theorem 18. *If real numbers $a_1, \ldots, a_n$ are positive, then the sequence (p_n) defined by the expansion* (61) *is logarithmically convex.*

The very simple proof of theorem 18 is based by OZEKI on the previously quoted assertion by SCHUR.

However, OZEKI's theorem 18 cannot be considered as original. For example, the statement of the same theorem is to be found in [23] already in 1962.

In connection with the same problems, see papers [21] — [30][1].

We stress particularly that in paper [31] (see also [8], p. 358) the expansion (60) was also considered and the following result was proved: If real numbers a_i $(i=1, \ldots, n)$ satisfy (60), then $(q_0=1)$

$$(q_{2r+1})^2 \leqq q_{2r}\, q_{2r+2} \qquad (r=0, 1, \ldots).$$

Here the positivity of numbers a_i was not assumed.

Let $(A_{n,k})$ be defined by

$$\binom{n}{k} A_{n,k} = \sum_{\substack{\alpha_i \in \{0,1\} \\ \alpha_1+\cdots+\alpha_n=k}} a_1^{\alpha_1} \cdots a_n^{\alpha_n} \tag{62}$$

and let $(B_{n,k})$ be defined by

$$\binom{n+k-1}{k} B_{n,k} = \sum_{\substack{\beta_i \in \mathbb{N} \\ \beta_1+\cdots+\beta_n=k}} a_1^{\beta_1} \cdots a_n^{\beta_n}; \tag{63}$$

In [7] OZEKI proved the following two theorems.

Theorem 19. *If the sequence (a_i) is positive and logarithmically convex, then*

$$A_{n-2,k}\, A_{n,k} \geqq (A_{n-1,k})^2 \qquad (n-2 \geqq k \geqq 1). \tag{64}$$

Theorem 20. *If the sequence (a_i) satisfies the assumptions of theorem* 19, *then*

$$B_{n-2,k}\, B_{n,k} \geqq (B_{n-1,k})^2 \qquad (n-2 \geqq k \geqq 1). \tag{65}$$

[1] T. POPOVICIU ([30], p. 2) stated that he obtained the same result (with $a_i>0$) independently of I. SCHUR. Besides, his result [29] was published before the book [20] which contains SCHUR's result as a private communication.

In [20], p. 52, we find that, if $a_i > 0\,(i=1, 2, \ldots)$, then inequalities

$$A_{n,k-2}\, A_{n,k} \leqq (A_{n,k-1})^2, \quad B_{n,k-2}\, B_{n,k} \geqq (B_{n,k-1})^2 \tag{66}$$

are valid.

The proof of theorems 19 and 20 is based on theorem 7. This proof will not be derived here. In connection with the symmetrc forms (62) and (63), i. e. inequalities (64), (65) and (66), see papers [21] — [28] containing far more general results for the symmetric forms.

4. Two inequalities

Mott [32] proved the following assertion:

If f is a nondecreasing, nonnegative integrable function on $[a, b]$, then, for every $x \in (a, b)$,

$$\frac{1}{x-a}\int_a^x f(u)\,\mathrm{d}u \leqq \frac{1}{b-a}\int_a^b f(u)\,\mathrm{d}u \leqq \int_x^b f(u)\,\mathrm{d}u. \tag{67}$$

In connection with (67) Ozeki [1] proved the following result:

Theorem 21. *If the function $x \mapsto p(x)$ is positive for $x \in [a, b]$ and if the function f is increasing on the same segment, then*

$$H(t_1, t_2) \leqq H(t_1, t_3) \leqq H(t_2, t_3) \qquad (t_1 < t_2 < t_3),$$

where

$$H(u, v) = \frac{F(u, v)}{G(u, v)}, \quad F(u, v) = \int_u^v p(x) f(x)\,\mathrm{d}x, \quad G(u, v) = \int_u^v p(x)\,\mathrm{d}x.$$

Ozeki's proof in [1] reduces to the investigations of derivatives of the function H with respect to u and v, where u and v are fixed numbers from $[a, b]$.

Inequality (67), as well as theorem 21, are considered in another sense in [33], [34] and [35] (see references given in [35]).

Ozeki [7] proved also an inequality relevant to the arithmetic means of real sequences. Nemely, he proved that the following assertion holds.

Theorem 22. *If $|\Delta^k a_n| \leqq M\,(n=1, 2, \ldots)$, then*

$$|\Delta^k A_n| \leqq \frac{M}{k+1} \quad (n=1, 2, \ldots),$$

where k is a fixed natural number and the sequence (A_n) is defined by (1).

The proof of this theorem is not quoted here because it is entirely analogous to the proof of theorem 3 of this paper.

5. A result for Bernstein's polynomials

Let us asume that the function f is defined and continuous on $[0, 1]$. Bernstein's polynomial $B_n(x; f)$ of order $n = 0, 1, \ldots$ of the function f is defined by

$$B_n(x; f) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right).$$

It is a well known fact that the sequence $B_n(x;f)$ uniformly converges towards $f(x)$, when $n \to +\infty$ under the quoted assumptions for the function f.

With reference to BERNSTEIN's polynomials OZEKI in [1] proved the following theorem (which is here presented for segment [0, 1], while in [1] thus theorem was proved for the segment $[a, b]$; they are equivalent to each other).

Theorem 23. *Let us asume that the function f is defined, positive and twice differentiable on* [0, 1]. *If the function f' is decreasing on* [0, 1], *then, for* $x \in [0, 1]$ *and* $n = 0, 1, \ldots$, *the following is true*

$$\text{(i) } B_{n+1}(x;f) > B_n(x;f) \text{ and (ii) } B_n(x;f) < f(x).$$

It is known that a differentiable function is concave (in the strict sense) if and only if the function f' is decreasing. Therefrom it follows that OZEKI's theorem concerns twice differentiable, positive concave functions.

OZEKI proved this theorem in 1965. As far as the conclusion (ii) of this theorem is concerned, it is an immediate consequence of the assertion under (i).

However, the first part of the theorem was already known in 1957. Namely, ARAMĂ in paper [36], from 1960, proved the following result.

Theorem 24. (a) *If f is a function convex, nonconcave, polynomial*[1], *nonconvex, concave on* [0, 1], *then the sequence $B_n(x;f)$ is decreasing, nonincreasing, stationary, nondecreasing, increasing, respectively.*

(b) *For an arbitrary continuous function f and* $\xi_1, \xi_2, \xi_3 \in [0, 1]$ *the following equality*[2]

$$B_{n+1}(x;f) - B_n(x;f) = -\frac{x(1-x)}{n(x+1)}[\xi_1, \xi_2, \xi_3; f]$$

is true.

As we see from [36] this theorem was published in Romanian in [49] already in 1957. From [36] we also learn that this theorem was rediscovered in 1959 by SCHOENBERG [50].

Thus OZEKI's theorem 23 contains also superfluous assumptions that $f > 0$ and that f'' exists for $x \in [0, 1]$ which is naturally a consequence of the proof given in [1].

In connection with theorem 24 we wish to quote the following. In [37] the oposite assertion is also proved: If the function f'' is continuous on [0, 1] and if $B_{n+1}(x;f) \leqq B_n(x;f)$ $(n = 0, 1, \ldots;\ x \in [0, 1])$ is valid, then f is convex on [0, 1]. In paper [38] among other things conditions on the function f are also given so that the inequality

$$\text{(68)} \qquad \Delta^2 B_n(x;f) = B_{n+2}(x;f) - 2B_{n+1}(x;f) + B_n(x;f) \geqq 0$$

is valid for all $x \in [0, 1]$ and $n = 1, 2, \ldots$. POPOVICIU in [39] gave some general results for interpolation polynomials analogous to previously proved results. In paper [40] the sequence $\frac{d}{dx} B_n(x;f)$ is investigated. Similar results

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[1] Continuous function f will be called polynomial if and only if $\Delta_h^2 f(x) = 0$ for all $h > 0$.

[2] For the definition of expression $[x_1, \ldots, x_{n+1}; f]$, see [8], p. 16.

are contained in paper **[41]**. MOLDOVAN in **[42]** weakened the assumptions under which a theorem inverse to theorem 24 is valid. Similar properties of some positive linear operators were investigated in **[43]** and **[45]**. KOSMAK **[44]** gave a characterization of nonconvex functions using BERNSTEIN's polynomials. Let us denote by S the class of all star-shaped functions on [0, 1] L. LUPAS in **[46]** showed the validity of the implication $f \in S \Rightarrow B_n(x; f) \in S$. HOROVA **[47]** weakened the conditions under which inequality (68) holds. In **[48]** results similar to the above are obtained for SZASZ-MIRAKYAN's operators.

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Added in proof

Dr A. LUPAŞ, acting as referee for this paper, informed us that there exist certain additions to the results exposed here. He gave us the following references:

1. W. B. TEMPLE: *Stieltjes integral representation of convex functions.* Duke Math. J. **21** (1954), 527—531.
2. D. D. STANCU: *On the monotonicity of the sequence formed by the first order derivatives of Bernstein polynomials.* Math. Z. **99** (1967), 46—51.
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4. И. И. ИБРАГИМОВ, А. Д ГАДЖИЕВ: *Об одной последовательности линейных положительных операторов.* ДАН (Doklady) СССР **193** (1970), 1222—1225.
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Judging by [1] it was W. B. TEMPLE who first obtained the results of theorem 24 (and hence the results of theorem 23) which are in connection with the behaviouı of BERNSTEIN's polynomials of convex functions. In other words theorem 24 was rediscovered by O. ARAMĂ and then by N. OZEKI.

D. D. STANCU [2] examined the behaviour of BERNSTEIN polynomials of convex functions. His paper presents a complement and a generalization of the previously ceted paper of O. ARAMĂ.

In paper [3] A. LUPAŞ and M. MÜLLER considered monotony and convexity of the sequence (G_n) of linear operators of the form

$$G_n(f;\, x)=\frac{x^{n+1}}{n}\int_0^{+\infty} e^{-xu}\, u^n f\left(\frac{n}{u}\right) du,$$

where f is a given function.

Paper [4] also contains certain results connected with linear operators defined on the cone of convex functions.

Certain properties of the n-th MEYER-KÖNIG and ZELLER operators of convex functions on [0, 1] were considered in [5], while [6] contains some new results for the operators introduced in [3].

Finally, paper [7] contains necessary and sufficient conditions which ensure that convexity with respect to a given ČEBYŠEV system remains invariant under a continuous linear operator $T: C[a, b] \to C[a, b]$.

*

Mr. G. KALAJDŽIĆ has some of OZEKI's papers elaborated in detail, completed and removed the shortcomings and thereby he had facilitated to the authors the composition of this exposition.

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