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# 611. SOME REMARKS ON THE EULERIAN FUNCTION

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*To Professor D. S. Mitrinović on his seventieth birthday.*

**1.** The EULERian function $H_n(x)$ can be defined by

$$\frac{1-x}{e^z-x}=\sum_{n=0}^{+\infty} H_n(x)\frac{z^n}{n!} \qquad (x\neq 1). \tag{1.1}$$

It follows from (1.1) that $H_n(x)$ is a rational function of $x$:

$$H_n(x)=\frac{A_n(x)}{(x-1)^n}, \tag{1.2}$$

where

$$A_n(x)=\sum_{k=1}^{n} A_{n,k}\,x^{k-1} \qquad (n\geqq 1). \tag{1.3}$$

The $A_{n,k}$ are called EULERIAN numbers. They are positive integers that satisfy the recurrence

$$A_{n+1,k}=(n-k+2)\,A_{n,k-1}+kA_{n,k} \tag{1.4}$$

and the symmetry relation

$$A_{n,k}=A_{n,n-k+1} \qquad (1\leqq k\leqq n). \tag{1.5}$$

The function $H_n(x)$ satisfies

$$H_n(x^{-1})=(-1)^n\,xH_n(x) \qquad (n\geqq 1) \tag{1.6}$$

and

$$xH_n(x)=\sum_{j=0}^{n}\binom{n}{j}H_j(x) \qquad (n\geqq 1). \tag{1.7}$$

It follows from (1.2) and (1.3) that (1.5) and (1.6) are equivalent.

FROBENIUS [6] has discussed properties of $H_n(x)$ at length with particular stress on arithmetic properties. For briefer treatments see [2] and [7].

The writer [3] has proved that

$$(y-x)\,(H(x)+H(y))^n=(1-x)\,H_n(y)-(1-y)\,H_n(x) \qquad (x\neq y), \tag{1.8}$$

where $(H(x)+H(y))^n=\sum_{j=0}^{n}\binom{n}{j}H_j(x)\,H_{n-j}(y)$. For $x=y$, we have

$$(x-1)\left(H_{n+1}(x)+H_n(x)\right)=x\left(H(x)+H(x)\right)^n \qquad (n\geqq 0). \tag{1.9}$$

We shall show that (1.9) characterizes $H_n(x)$. However this is not true of (1.8). We shall show that if $\{f_n(x)\}$ is a sequence that satisfies

$$(y-x)\left(f(x)+f(y)\right)^n=(1-x)\,f_n(y)-(1-y)f_n(x) \qquad (n\geqq 0) \tag{1.10}$$

and $F=F(x,\ z)=\sum_{n=0}^{+\infty}f_n(x)\frac{z^n}{n!}$ $(f_0(x)=1)$ then

$$F(x,\ z)=\frac{1-x}{C(z)-x}, \tag{1.11}$$

where

$$C(z)=\sum_{n=0}^{+\infty}c_n\frac{z^n}{n!} \qquad (c_0=1). \tag{1.12}$$

It follows from (1.11) and (1.12) that

$$f_n(x)=\frac{C_n(x)}{(x-1)^n},\quad C_n(x)=\sum_{k=1}^{n}C_{n,k}\,x^{k-1} \qquad (n\geqq 1),$$

where the coefficients $C_{n,k}$ are determined by $C(z)$. Moreover, $C_{n,k}$ satisfies the symmetry condition

$$C_{n,k}=C_{n,n-k+1} \qquad (1\leqq k\leqq n) \tag{1.13}$$

if and only if $C(z)$ satisfies

$$C(z)\,C(-z)=1. \tag{1.14}$$

This condition is equivalent to

$$C(z)=\frac{\Phi(z)}{\Phi(-z)},\quad \Phi(z)=1+\sum_{n=1}^{+\infty}d_n\frac{z^n}{n!}. \tag{1.15}$$

In the EULERian case, if we put [4]:

$$A(r,\ s)=A_{r+s+1,\,r+1}=A_{r+s+1,\,s+1}=A(s,\ r),$$

the generating function (1.1) becomes

$$\sum_{r,\,s=0}^{+\infty}A(r,\ s)\frac{x^r\,y^s}{(r+s+1)!}=\frac{e^x-e^y}{xe^y-ye^x}. \tag{1.16}$$

Similarly, if we put $C(r,s)=C_{r+s+1,\,r+1}=C_{r+s+1,\,s+1}=C(s,r)$, the generating function (1.11) becomes

$$\sum_{r,\,s=0}^{+\infty}C(r,\ s)\frac{x^r\,y^s}{(r+s+1)!}=\frac{\Phi(x-y)-\Phi(y-x)}{x\,\Phi(y-x)-y\,\Phi(y-x)}. \tag{1.17}$$

We remark that the functional equation (1.14) as well as the generating function (1.17) have occurred in [5] in connection with the following combinatorial problem: the enumeration of pairs of *amicable* permutations.

The writer [1] has defined „degenerate" EULERian numbers by means of

$$\frac{1-x}{1-x\,(1+\lambda z\,(1-x))^{\mu}}=1+\sum_{n=1}^{+\infty}\frac{z^n}{n!}\sum_{k=1}^{n}A_{n,k}(\lambda)\,x^k, \tag{1.18}$$

where $\lambda\mu=1$. This suggests that we put

$$\frac{1-x}{1-x(1+\lambda z)^{\mu}}=1+x\sum_{n=1}^{+\infty}\frac{z^n}{n!}H_n(\lambda,\,x) \qquad (\lambda\mu=1). \tag{1.19}$$

It is then easy to show that

$$xH_n\left(-\lambda,\,\frac{1}{x}\right)=(-1)^n H_n(\lambda,\,x) \tag{1.20}$$

and

$$(x-y)\big(H(\lambda,\,x)+H(\lambda,\,y)\big)^n=x(1-y)\,H_n(\lambda,\,x)-y\,(1-x)\,H_n(\lambda,\,y) \quad (x\neq y). \tag{1.21}$$

Note that the notation in (1.18) and (1.19) is somewhat different from that in the EULERian case.

In view of (1.21) we consider the problem of characterizing sequences $\{f_n(\lambda,\,x)\}$ sush that $f_0(\lambda,\,x)=1$ and

$$(x-y)\,(f(\lambda,\,x)+f(\lambda,\,y))^n=x\,(1-y)f_n(\lambda,\,x)-y\,(1-x)f_n(\lambda,\,y) \qquad (x\neq y). \tag{1.22}$$

The results are similar to those in the simpler case (1.10). In particular we show that

$$\sum_{n=0}^{+\infty}f_n(\lambda,\,x)\frac{z^n}{n!}=\frac{1-x}{1-xC(\lambda,\,z)}, \tag{1.23}$$

where $C(\lambda,\,z)$ is a power series in $z$, $C(\lambda,\,0)=1$. Moreover $f_n(\lambda,\,x)$ satisfies

$$xf_n\left(-\lambda,\,\frac{1}{x}\right)=(-1)^n f_n(\lambda,\,x) \tag{1.24}$$

if and only if

$$C(\lambda,\,z)\,C(-\lambda,\,-z)=1 \tag{1.25}$$

or equivalently

$$C(\lambda,\,z)=\Phi(\lambda,\,z)/\Phi(-\lambda,\,-z). \tag{1.26}$$

It can be verified that (1.23) implies the symmetrical generating function (compare (1.17))

$$\sum_{r,\,s=0}^{+\infty}C(r,\,s,\,\lambda)\frac{x^r y^s}{(r+s+1)!}=\frac{\Phi(-\lambda,\,x-y)-\Phi(\lambda,\,y-x)}{x\Phi(\lambda,\,y-x)-y\,\Phi(-\lambda,\,x-y)}, \tag{1.27}$$

where $f_n(\lambda, x) = (1-x)^{-n} \sum_{k=1}^{n} C_{n,k}(\lambda)x^k$ $(n \geqq 1)$ and

$$C(r, s, \lambda) = C_{r+s+1, r+1}(\lambda) = C_{r+s+1, s+1}(-\lambda) = C(s, r, -\lambda).$$

**2.** We show first that $H_n(x)$ is characterized by (1.9). More precisely let $\{f_n(x)\}$ be a sequence of functions, $f_0(x) = 1$, that satisfy

$$(2.1) \qquad (x-1)(f_{n+1}(x) + f_n(x)) = x(f(x) + f(x))^n \quad (n = 0, 1, 2, \ldots).$$

Put

$$(2.2) \qquad F = F(x, z) = \sum_{n=0}^{+\infty} f_n(x) \frac{z^n}{n!}.$$

Then

$$\sum_{n=0}^{+\infty} (f(x) + f(x))^n \frac{z^n}{n!} = \sum_{r, s=0}^{+\infty} f_r(x) f_s(x) \frac{z^{r+s}}{r!\, s!} = F^2.$$

On the other hand

$$\sum_{n=0}^{+\infty} (f_{n+1}(x) + f_n(x)) \frac{z^n}{n!} = F_z + F \qquad \left(F_z \equiv \frac{\partial F}{\partial z}\right).$$

Thus (2.1) gives

$$(2.3) \qquad (x-1)(F_z + F) = xF^2.$$

Solving the differential equation (2.3) we get $F(x, z) = \frac{1-x}{C(x)\, e^z - x}$, where $C(x)$ is some function of $x$. Since $F(x, 0) = f_0(x) = 1$, it follows that $C(x) = 1$. Hence

$$F(x, z) = \frac{1-x}{e^z - x}, \quad f_n(x) = H_n(x).$$

This proves the following theorem.

**Theorem 1.** *Let $\{f_n(x)\}$ be a sequence of functions, $f_0(x) = 1$, that satisfy*

$$(2.4) \qquad (x-1)(f_{n+1}(x) + f_n(x)) = x(f(x) + f(x))^n \qquad (n = 0, 1, 2, \ldots).$$

*Then*

$$(2.5) \qquad f_n(x) = H_n(x) \qquad (n = 0, 1, 2, \ldots).$$

We now consider a sequence of functions $\{f_n(x)\}$ that satisfy

$$(2.6) \qquad (y-x)(f(x) + f(y))^n = (1-x) f_n(y) - (1-y) f_n(x) \qquad (x \neq y)$$

for $n = 0, 1, 2, \ldots$ Note that for $n = 0$, (2.6) gives $f_0(x) = 1$.

As in the previous case we define $F = F(x, z)$ by means of (2.2). Then it is easily verified that (2.6) implies

$$(2.7) \qquad (y-x) F(x, z) F(y, z) = (1-x) F(y, z) - (1-y) F(x, z).$$

Divide both sides of (2.7) by $y-x$ and let $y \to x$. We find that

$$F^2(x, z) = (1-x) F_x(x, z) + F(x, z). \tag{2.8}$$

It follows from (2.8) that

$$F(x, z) = \frac{1-x}{C(z)-x}, \tag{2.9}$$

where $C(z)$ is now a function of $z$. Since $F(x, 0) = f_0(x) = 1$, it is clear that $C(0) = 1$.

Substituting from (2.9) in (2.7) we get

$$\frac{(y-x)(1-x)(1-y)}{(C(z)-x)(C(z)-y)} = \frac{(1-x)(1-y)}{C(z)-y} - \frac{(1-x)(1-y)}{C(z)-x},$$

an identity in $x, y$ and $C(z)$. Hence $C(z)$ is arbitrary except for $C(0) = 1$.

This completes the proof of the following theorem.

**Theorem 2.** *Let $\{f_n(x)\}$ be a sequence of functions that satisfy*

$$(y-x)(f(x)+f(y))^n = (1-x) f_n(y) - (1-y) f_n(x) \qquad (y \neq x) \tag{2.10}$$

*for $n = 0, 1, 2, \ldots$ Then*

$$\sum_{n=0}^{+\infty} f_n(x) \frac{z^n}{n!} = \frac{1-x}{C(z)-x}, \tag{2.11}$$

*where $C(z)$ is an arbitrary function of $z$, $C(0) = 1$. Conversely,* (2.11) *implies* (2.10).

We remark that it follows from (2.10) that

$$(f(x)+f(y)+f(z))^n = \frac{(1-y)(1-z)}{(x-y)(x-z)} f_n(x) + \frac{(1-z)(1-x)}{(y-z)(y-x)} f_n(y) + \frac{(1-x)(1-y)}{(z-x)(z-y)} f_n(z) \tag{2.12}$$

and similarly for a larger number of variables. The general formula of this kind can be proved most easily by expressing the product

$$\frac{1-x_1}{C(z)-x_1} \, \frac{1-x_2}{C(z)-x_2} \cdots \frac{1-x_n}{C(z)-x_n}$$

as a sum of partial fractions.

**3.** We shall now assume that $C(z)$ is analytic in the neighborhood of the origin:

$$C(z) = \sum_{n=0}^{+\infty} c_n \frac{z^n}{n!}, \quad c_0 = 1. \tag{3.1}$$

Put

$$(C(z)-1)^k = \sum_{n=k}^{+\infty} c_n^{(k)} \frac{z^n}{n!} \qquad (k = 1, 2, \ldots). \tag{3.2}$$

Then

$$\frac{1-x}{C(z)-x}=\frac{1-x}{(C(z)-1)+(1-x)}=1+\sum_{k=1}^{+\infty}(x-1)^{-k}(C(z)-1)^k$$

$$=1+\sum_{k=1}^{+\infty}(x-1)^{-k}\sum_{n=k}^{+\infty}c_n^{(k)}\frac{z^n}{n!}=1+\sum_{n=1}^{+\infty}\frac{z^n}{n!}\sum_{k=1}^{n}c_n^{(k)}(x-1)^{-k}$$

$$=1+\sum_{n=1}^{+\infty}(x-1)^{-n}\frac{z^n}{n!}\sum_{k=1}^{n}c_n^{(k)}(x-1)^{n-k}.$$

Comparison with (2.11) gives

$$f_n(x)=(x-1)^{-n}\sum_{k=1}^{n}c_n^{(k)}(x-1)^{n-k} \qquad (n=1,2,\ldots). \tag{3.3}$$

Hence, for $n\geqq 1$, $f_n(x)$ is a rational function of $x$. We may put

$$f_n(x)=\frac{C_n(x)}{(x-1)^n}, \quad C_n(x)=\sum_{k=1}^{n}C_{n,k}x^{k-1} \qquad (n\geqq 1). \tag{3.4}$$

Note that

$$C_{nn}=c_n^{(1)}=c_n. \tag{3.5}$$

In view of (1.5) it is of interest to determine whether $C_{n,k}$ satisfies the symmetry relation

$$C_{n,k}=C_{n,n-k+1} \qquad (1\leqq k\leqq n) \tag{3.6}$$

for appropriate $C(z)$. The relation

$$f_n(-x)=(-1)^n x f_n(x) \qquad (n\geqq 1) \tag{3.7}$$

is equivalent to (3.6).

Making use of (3.7), we have

$$F(x^{-1},z)=1+x\sum_{n=1}^{+\infty}(-1)^n f_n(x)\frac{z^n}{n!}=1+x\big(F(x,-z)-1\big),$$

so that

$$F(x^{-1},z)=1-x+xF(x,-z). \tag{3.8}$$

Thus by (2.11)

$$\frac{1-x^{-1}}{C(z)-x^{-1}}=1-x+\frac{x(1-x)}{C(-z)-x}.$$

Simmplifying, we get

$$C(z)C(-z)=1. \tag{3.9}$$

The steps are reversible and therefore (3.9) is a necessary and sufficient condition that (3.6) is satisfied.

The condition (3.9) is obviously satisfied by $C(z)=e^z$. More generally it is satisfied if

$$C(z)=\frac{\Phi(z)}{\Phi(-z)}, \tag{3.10}$$

where

$$\Phi(z)=1+\sum_{n=1}^{+\infty} d_n \frac{z^n}{n!}, \tag{3.11}$$

is any function analytic about the origin such that $\Phi(0)=1$.

Conversely (3.9) implies (3.10). To prove this put $C(z)=E(z)+E_1(z)$, where $E(z)$ is even and $E_1(z)$ is odd. Then

$$C(z)\,C(-z)=(E(z)+E_1(z))\,(E(z)-E_1(z)),$$

so that, by (3.9),

$$E^2(z)-E_1{}^2(z)=1,\quad E_1(z)=\sqrt{E^2(z)-1}. \tag{3.12}$$

Consider

$$\Phi(z)=\Phi_0(z)\left(1+\sqrt{\frac{E(z)-1}{E(z)+1}}\right)=\Phi_0(z)\,\frac{\sqrt{E(z)+1}+\sqrt{E(z)-1}}{\sqrt{E(z)+1}}$$

where $\Phi_0(z)$ is an arbitrary even function, $\Phi_0(0)=1$. Then

$$\Phi(-z)=\Phi_0(z)\,\frac{\sqrt{E(z)+1}-\sqrt{E(z)-1}}{\sqrt{E(z)+1}},$$

$$\frac{\Phi(z)}{\Phi(-z)}=\frac{\sqrt{E(z)+1}+\sqrt{E(z)-1}}{\sqrt{E(z)+1}-\sqrt{E(z)-1}}=\frac{1}{2}\left(\sqrt{E(z)+1}+\sqrt{E(z)-1}\right)^2$$

$$=E(z)+\sqrt{E^2(z)-1}=C(z).$$

This completes the proof of

**Theorem 3.** *Let the sequence* $\{f_n(x)\}_1^{+\infty}$ *satisfy* (2.10). *It will also satisfy*

$$f_n(-x)=(-1)^n x f_n(x) \qquad (n=1,2,\dots) \tag{3.13}$$

*if and only if the coefficient* $C_{n,k}$ *satisfies*

$$C_{n,k}=C_{n,n-k+1} \qquad (1\le k\le n). \tag{3.14}$$

*The condition* (3.13) *is satisfied if and only if* $C(z)=1+\sum_{n=1}^{+\infty} c_n \frac{z^n}{n!}$ *satisfies*

$$C(z)\,C(-z)=1 \tag{3.15}$$

*or equivalently*

$$C(z)=\frac{\Phi(z)}{\Phi(-z)},\quad \Phi(z)=1+\sum_{n=1}^{+\infty} d_n \frac{z^n}{n!}. \tag{3.16}$$

If we replace $z$ by $(x-1)z$ the generating function becomes

$$\sum_{r,s=0}^{+\infty} C(r,s)\frac{x^r y^s}{(r+s+1)!}=\frac{\Phi(x-y)-\Phi(y-x)}{x\Phi(y-x)-y\Phi(x-y)}, \tag{3.17}$$

where $C(r,s)=C_{r+s+1,r+1}=C_{r+s+1,s+1}=C(s,r)$.

We remark that if $G=G(x,y)=\dfrac{\Phi(x-y)-\Phi(y-x)}{x\Phi(y-x)-y\Phi(x-y)}$, it can be verified that

$$G_x+G_y=G^2. \tag{3.18}$$

Also

$$G+xG_x+yG_y=(x-y)^2\left(\Phi(x-y)\Phi'(y-x)+\Phi(y-x)\Phi'(x-y)\right),$$
$$1+(x+y)G+xyG^2=(x-y)^2\Phi(x-y)\Phi(y-x).$$

It follows that

$$\sum_{r,s=0}^{+\infty} C(r,s)\frac{x^r y^s}{(r+s)!}=\frac{(x-y)^2\left(\Phi(x-y)\Phi'(y-x)+\Phi(y-x)\Phi'(x-y)\right)}{(x\Phi(y-x)-y\Phi(x-y))^2}.$$

Only in the EULERian case $(C(r,s)=A(r,s))$ do we have

$$\sum_{r,s=0}^{+\infty} C(r,s)\frac{x^r y^s}{(r+s)!}=(1+xG(x,y))(1+yG(x,y)).$$

It follows from (3.17) that

$$\sum_{r,s=0}^{+\infty} C(r,s)\frac{x^{r+s}}{(r+s+1)!}=\frac{2\Phi'(0)}{1-2x\Phi'(0)}=\frac{c_1}{1-c_1x},$$

so that $\sum_{r+s+n} C(r,s)=c_1^{n+1}(n+1)!$ or, what is the same,

$$\sum_{k=1}^{n} C_{n,k}=c_1^n n! \tag{3.20}$$

**4.** We now consider the „degenerate" EULERian number $A_{n,k}(\lambda)$ defined by

$$\frac{1-x}{1-x(1+\lambda z(1-x))^{\mu}}=1+\sum_{n=1}^{+\infty}\frac{z^n}{n!}\sum_{k=1}^{n}A_{n,k}(\lambda)x^k \qquad (\lambda\mu=1). \tag{4.1}$$

It follows that $A_{n,k}(\lambda)$ is a polynomial in $\lambda$. Put

$$xH_n(\lambda,x)=(1-x)^{-n}\sum_{k=1}^{n}A_{n,k}(\lambda)x^k \qquad (n\geqq 1),$$

so that (4.1) becomes

$$\frac{1-x}{1-x(1+\lambda z)^{\mu}}=\sum_{n=0}^{+\infty}\frac{z^n}{n!}H_n(\lambda,x). \tag{4.2}$$

It folows from (4.2) that $H_n(\lambda, x)$ satisfies

(4.3) $$x(1-y)H_n(\lambda, x)-y(1-x)H_n(\lambda, y)=(x-y)\big(H(\lambda, x)+H(\lambda, y)\big)^n \qquad (x\neq y)$$

where $H_0(\lambda, x)=1$. Differentiation of (4.2) with respect to $z$ yields

$$x(1-x)(1+\lambda x)\sum_{n=0}^{+\infty}\frac{z^n}{n!}H_{n+1}(\lambda)=\left(\frac{1-x}{1-x(1+\lambda z)^\mu}\right)^2-\frac{(1-x)^2}{1-x(1+\lambda z)^\mu}.$$

Comparing coefficients of $z^n$, we get

(4.4) $$(1-x)H_{n+1}(\lambda, x)+(1-x)(1+n\lambda)H_n(\lambda, x)=\big(H(\lambda, x)+H(\lambda, x)\big)^n.$$

To get the quasi-symmetry property we replace $\lambda$ by $-\lambda$ and $x$ by $x^{-1}$ in (4.2). This gives

(4.5) $$xH_n\left(-\lambda, \frac{1}{x}\right)=(-1)^n H_n(\lambda, x) \qquad (n\geqq 1).$$

Since $H_n(\lambda, x)=(1-x)^{-n}\sum_{k=1}^{n}A_{n,k}(\lambda)x^k$, (4.5) is equivalent to

(4.6) $$A_{n,n-k+1}(\lambda)=A_{n,k}(-\lambda) \qquad (1\leqq k\leqq n).$$

Now let $\{f_n(\lambda, x)\}_0^{+\infty}$ be a sequence such that $f_0(\lambda, x)=1$ and

(4.7) $$(1-x)f_{n+1}(\lambda, x)+(1-x)(1+n\lambda)f_n(\lambda, x)$$
$$=\big(f(\lambda, x)+f(\lambda, x)\big)^n \qquad (n=0, 1, 2, \dots).$$

Put

(4.8) $$F\equiv F(\lambda, x, z)=\sum_{n=0}^{+\infty}f_n(\lambda, x)\frac{z^n}{n!}.$$

Since $F_z(\lambda, x, z)=\sum_{n=0}^{+\infty}f_{n+1}(\lambda, x)\frac{z^n}{n!}$ and $zF_z(\lambda, x, z)=\sum_{n=0}^{+\infty}nf_n(\lambda, x)\frac{z^n}{n!}$, it follows from (4.7) that

(4.9) $$(1-x)(1+\lambda z)F_z=F^2-(1-x)F.$$

The general solution of (4.9) is

$$\frac{F-1+x}{F}=K(1+\lambda z)^\mu.$$

Since $F(\lambda, x, 0)=1$, $K=x$ and therefore

$$F(\lambda, x, z)=\frac{1-x}{1-x(1+\lambda z)^\mu}.$$

We may state:

**Theorem 4.** *Let* $\{f_n(\lambda, x)\}_0^{+\infty}$ *denote a sequence such that* $f_0(\lambda, x)=1$ *and*

$(1-x)f_{n+1}(\lambda, z)+(1-x)(1+n\lambda)f_n(\lambda, x)=f(\lambda, x)+f(\lambda, x))^n \quad (n=0, 1, 2, \ldots).$

*Then*

$$f_n(\lambda\, x)=H_n(\lambda, x) \qquad (n=0=0, 1, 2, \ldots).$$

We now consider a sequence $\{f_n(\lambda, x)_0^{+\infty}$ such that

$$(4.10) \qquad x(1-y)f_n(\lambda, x)-y(1-x)f_n(\lambda, y)$$
$$=(x-y)(f(\lambda, x)+f(\lambda, y))^n(x\neq y); \quad n=0, 1, 2, \ldots$$

Again define $F(\lambda, x, z)$ by (4.8). Then (4.10) implies

$$(4.11) \quad x(1-y)F(\lambda, x, z)-y(1-x)F(\lambda, y, z)=(x-y)F(\lambda, x, z)F(\lambda, y, z).$$

Divide both sides of (4.11) by $x-y$ and let $y\to x$. We get

$$(4.12) \qquad F^2(\lambda, x, z)=F(\lambda, x, z)+x(1-x)F_x(\lambda, x, z).$$

The general solution of (4.12) is

$$(4.13) \qquad F(\lambda, x, z)=\frac{1-x}{1-xC(\lambda, z)}.$$

Since $F(\lambda, x, 0)=1$, $C(\lambda, 0)=1$.

This proves

**Theorem 5.** *Let* $\{f_n(\lambda, x\}_0^{+\infty}$ *denote a sequence such that* $f_0(\lambda, x)=1$ *and*

$$x(1-y)f_n(\lambda, x)-y(1-x)f_n(\lambda, y)$$
$$=(x-y)(f(\lambda, x)+f(\lambda, y))^n \quad (x\neq y;\ n=0, 1, 2, \ldots).$$

*Then*

$$(4.14) \qquad \sum_{n=0}^{+\infty} f_n(\lambda, x)\frac{z^n}{n!}=\frac{1-x}{1-xC(\lambda, z)},$$

*where*

$$(4.15) \qquad C(\lambda, z)=1+\sum_{n=1}^{\infty} c_n(\lambda)z^n.$$

**5.** We now require that $f_n(\lambda, x)$ satisfy

$$(5.1) \qquad xf_n\left(-\lambda, \frac{1}{x}\right)=(-1)^n f_n(\lambda, x) \qquad (n\geqq 1).$$

It follows that

$$F(\lambda, x, z)=1+x\sum_{n=1}^{+\infty}(-1)^n f_n\left(-\lambda, \frac{1}{x}\right)\frac{z^n}{n!}=1-x+x\sum_{n=0}^{+\infty}(-1)^n f_n\left(-\lambda, \frac{1}{x}\right)\frac{z^n}{n!},$$

so that

(5.2) $$F(\lambda, x, z) = 1 - x + xF\left(-\lambda, \frac{1}{x}, -z\right).$$

Hence by (4.14) and (5.2),

$$\frac{1-x}{1-xC(\lambda, z)} = 1 - x + \frac{x(1-x^{-1})}{1-x^{-1}C(-\lambda, -z)}.$$

Simplifying, we get

(5.3) $$C(\lambda, z)C(-\lambda, -z) = 1.$$

We shall show that (5.3) is equivalent to

(5.4) $$C(\lambda, z) = \frac{\Phi(\lambda, z)}{\Phi(-\lambda, -z)}.$$

It follows from (5.3) that

(5.5) $$\log C(\lambda, z) + \log C(-\lambda, -z) = 0.$$

Thus we may put $\log C(\lambda, z) = \sum_{r=0}^{+\infty} \sum_{s=1}^{+\infty} c_{rs} \lambda^r z^s$; by (4.15) the lower limit for $s \geqq 1$. Hence, by (5.5), $\sum_{r,s} (1 + (-1)^{r+s}) c_{rs} \lambda^r z^s = 0$, so that

(5.6) $$c_{rs} = 0 \quad (r + s \equiv 0 \pmod 2).$$

Let $P(\lambda, z)$ denote a power series in $\lambda$ and $z$: $P(\lambda, z) = \sum_{r,s} p_{rs} \lambda^r z^s$ and consider the functional equation

(5.7) $$P(\lambda, z) - P(-\lambda, z) = \log C(\lambda, z).$$

This is equivalent to

(5.8) $$(1 - (-1)^{r+s}) p_{rs} = c_{rs}.$$

For $r + s \equiv 0 \pmod 2$, (5.8) is satisfied automatically because of (5.6); for $r + s \equiv 1 \pmod 2$, $p_{rs}$ is uniquely determined by (5.8). Hence, for $\Phi(\lambda, z) = \exp P(\lambda, z)$, we get (5.4).

The series $\Phi(\lambda, z)$ is of course not uniquely determined; the above proof indicates that $\Phi(\lambda, z)$ may be multipled by an arbitrary series of the form $\sum_{r+s=0\,(2)} d_{rs} \lambda^r z^s$.

Summing up, we state

**Theorem 6.** *Let the sequence* $\{f_n(\lambda, x)\}_1^{+\infty}$ *satisfy* (4.10). *It will also satisfy*

(5.9) $$xf_n\left(-\lambda, \frac{1}{x}\right) = (-1)^n f_n(\lambda, x) \qquad (n = 1, 2, \ldots)$$

*if and only if*

(5.10) $$C(\lambda, z)\, C(-\lambda, -z) = 1$$

*or equivalently*

(5.11) $$C(\lambda, z) = \Phi(\lambda, z)/\Phi(-\lambda, -z),$$

*where* $\Phi(\lambda, z)$ *may be assumed to be of the form* $\Phi(\lambda, z) = \sum_{r,s} a_{rs} \lambda^r z^s$ *and*

(5.12) $$a_{rs} = 0 \ (r+s>0;\ r+s \equiv 0 \pmod 2).$$

As an example, for $C(\lambda, z) = (1+\lambda z)^{\mu}$, we may take $\Phi(\lambda, z) = (1+\lambda z)^{\mu/2}$, although this choice of $\Phi(\lambda, z)$ does not satisfy (5.12).

By (4.14) and (5.11) we have

(5.13) $$1 + \sum_{n=1}^{+\infty} \frac{z^n}{n!} \sum_{k=1}^{n} C_{n,k}(\lambda) x^k = \frac{1-x}{1-xC(\lambda, z-xz)}$$

$$= \frac{(1-x)\, \Phi(-\lambda, xz-x)}{\Phi(-\lambda, xz-x) - x\, \Phi(\lambda, z-xz)},$$

where

$$(1-x)^n f_n(\lambda, x) = \sum_{k=1}^{n} C_{n,k}(\lambda)\, x^k.$$

Replacing $x$ by $xz^{-1}$ in (5.13) we get after a little manipulation

(5.14) $$\sum_{r,s=0}^{+\infty} C(r, s, \lambda) \frac{x^r y^s}{(r+s+1)!} = \frac{\Phi(\lambda, y-x) - \Phi(-\lambda, x-y)}{y\, \Phi(-\lambda, x-y) - x\, \Phi(\lambda, y-x)},$$

where

(5.15) $$C(r, s, \lambda) = C_{r+s+1,\, r+1}(\lambda) = C_{r+s+1,\, s+1}(-\lambda) = C(s, r, -\lambda).$$

It is evident frum (4.2) that $H_n(\lambda, x)$ is a polynomial in $\lambda$ of degre $\leqq n$. More precisely it follows from

$$H_n(\lambda, x) = (1-x)^{-n} A_n(\lambda, x), \qquad A_n(\lambda, x) = \sum_{k=1}^{n} A_{n,k}(\lambda)\, x^k,$$

that $A_n(\lambda, x)$ is a polynomial of degree $\leqq n$ in $x$ and of degree $\leqq n-1$ in $\lambda$. Moreover it is proved in **[1]** that $A_{n,k}(\lambda)$ is a polynomial in $\lambda$ of degree $n-1$ for $k = 1, \ldots, n$.

In the more general case of $f_n(\lambda, x)$, we can again assert that

$$(1-x)^n f_n(\lambda, x)$$

is a polynomial of degree $\leqq n$ in $x$. As for the parameter $\lambda$, if we put

$$C(\lambda, z) = 1 + \sum_{n=1}^{+\infty} c_n(\lambda) \frac{z^n}{n!} \tag{5.16}$$

and assume that $c_n(\lambda)$ is a polynomial of degree $\leqq n-1$ in $\lambda$, then this is also true of $C_{n,k}(\lambda)$. Also it follows from (4.14) and (5.16) that

$$C_{n,1}(\lambda) = c_n(\lambda).$$

Hence, if (5.9) is assumed to hold, we get

$$C_{n,n}(\lambda) = C_{n,1}(-\lambda) = c_n(-\lambda).$$

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