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603. EQUI-AFFINITIES IN THREE-DIMENSIONAL SPACE

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Dedicated to D. S. Mitrinović on the occasion of his seventieth birthday.

Cher ami, nous ne nous sommes jamais rencontrés. Je ne connais ni votre voix, ni vos circonstances personnelles, ni votre conception de la vie. Mais pendant plusieurs années notre communication par écrit a transporté de Belgrade à Delft, aller et retour, beaucoup de théorèmes, de questions, de propositions et surtout d'inégalités. Cette correspondance a atteint son apogée lorsque vous m'avez invité à joindre votre cénacle de mathématiciens doués et actifs qui, sous votre direction, ont compilé une collection (je m'exuse: *la* collection) systématiquement arrangée, des inégalités de la géométrie plane, publiée en 1969 comme *Geometric Inequalities.*

Notre amitié scientifique a pour moi une valeur très importante; plusieurs fois mes pensées se sont envolées vers vous et ma fantaisie vous a placé a votre bureau, entouré par des centaines de livres et de journaux mathématiques, ou en discutant avec vos élèves qui sont devenus vos collaborateurs et vos amis.

Cher MITRINOVIĆ, en vous remerciant pour tout ceque vous avez réalisé dans notre science, j'ai tâché d'exprimer mes sentiments chaleureux pour votre personne non pas au moyen de la *lingua franca* de nos jours, mais par l'instrument de la culture latine pour laquelle vous avez une si profonde admiration.

1. Introduction. In the plane and in three-dimensional space the following theorem is well-known: any Euclidean displacement may be written as the product of two line reflections. It can be applied for instance to develop an elegant method to study three positions theory in Euclidean kinematics. The reflection has an analog in affine geometry. For the affine space such a transformation $R(m; U)$ is defined as follows. Let a line m, the mirror, and a plane U, the direction plane, be given; m and U are not parallel. If P is an arbitrary point, U' the plane through P parallel to U, S its intersection with m, then the point P' corresponding to P is on the ray PS, such that $PS+SP'=0$. Obviously $R^2=I$, the unit transformation; furthermore R is volume-preserving. The product $T=R_2 R_1$ of two reflections is an affine, volume-preserving transformation, an *equi-affinity.* The question arises whether any equi-affinity can be factorized as the product of two reflections. RUTH STRUIK [1] studied this problem long ago by the methods of synthetic geometry. Her interesting and somewhat surprising results are: the property is valid for the analogous problem in the plane, but it does not hold in space. She added the positive theorem: an equi-affinity in space is always the product of three reflections.

In the following note we consider, by analytical means, all possible products $T = R_2 R_1$, with $R_i = (m_i;\ U_i)$, $i = 1, 2$ and study the properties of T. It will be seen that the set T does not cover all equi-affinities, which confirms RUTH STRUIK's statement. We distinguish four cases: I. the mirrors m_1, m_2 are skew lines; II. m_1 and m_2 intersect; III. m_1 and m_2 are parallel; IV. m_1 and m_2 coincide.

2. The mirrors m_1, m_2 are skew lines. We introduce an affine coordinate system $OXYZ$ as follows. We take OX and OY parallel to m_1 and m_2 respectively. If the direction planes U_1, U_2 have a line of intersection s, not parallel to OXY, we take as OZ the (unique) line, parallel to s and intersecting m_1 and m_2 at E_1 and E_2; let O be the midpoint of $E_1 E_2$. The direction plane U_i, which we can always take through O, has the equation $A_i x + B_i y = 0$, $(i = 1, 2)$. As U_i is not parallel to m_i we have $A_1 \neq 0$, $B_2 \neq 0$, hence we may suppose $A_1 = B_2 = 1$.

Summing up we have

$$(2.1) \qquad \begin{aligned} &m_1:\ y = 0,\ z = d; \qquad &U_1:\ x + B_1 y = 0,\\ &m_2:\ x = 0,\ z = -d; \qquad &U_2:\ A_2 x + y = 0, \end{aligned}$$

with $d \neq 0$.

We derive the formulas of the reflection $R_1(m_1;\ U_1)$. Let $P = (x_0, y_0, z_0)$, then U_1' has the equation $x + B_1 y = x_0 + B_1 y_0$; for $S_1 = (x_1, y_1, z_1)$ we have $x_1 = x_0 + B_1 y$, $y_1 = 0$, $z_1 = d$. The ray PS_1 is represented by $x = (x_0 + B_1 y_0 + \lambda x_0)/(1 + \lambda)$, $y = \lambda y_0/(1 + \lambda)$, $z = (d + \lambda z_0)/(1 + \lambda)$; the points P, S_1 and the point at infinity correspond to $\lambda = \infty$, $\lambda = 0$ and $\lambda = -1$. In view of $PS_1 + S_1 P = 0$, P' corresponds to $\lambda = -\frac{1}{2}$. Hence we obtain for the reflection R_1:

$$(2.2) \qquad x' = x + 2 B_1 y, \quad y' = -y, \quad z' = -z + 2d.$$

In the same way we have for R_2:

$$(2.3) \qquad x' = -x, \quad y' = 2 A_2 x + y, \quad z' = -z - 2d.$$

From (2.2) and (2.3) the equi-affinity $T = R_2 R_1$ follows:

$$(2.4) \qquad x' = -x - 2 B_1 y, \quad y' = 2 A_2 x + (4 A_2 B_1 - 1) y, \quad z' = z - 4d.$$

In discussing the type of T we shall restrict ourselves to the determination of its invariant *points*. As $d \neq 0$ there are no finite invariant points.

Those in the plane V at infinity satisfy the linear homogeneous equations

$$(2.5) \qquad x(1 + \lambda) + 2 B_1 y = 0, \quad 2 A_2 x + (4 A_2 B_1 - 1 - \lambda) y = 0, \quad z(1 - \lambda) = 0,$$

which have only a solution if

$$(2.6) \qquad (\lambda - 1)\{\lambda^2 - 2(2 A_2 B_1 - 1)\lambda + 1\} = 0.$$

The discriminant of the quadratic factor is $D = 4 A_2 B_1 (A_2 B_1 - 1)$.

As U_1 and U_2 are not parallel we have $A_2 B_1 \neq 1$. If $A_2 B_1 \neq 0$, (that means: U_1 is not parallel to m_2, U_2 not parallel to m_1) we have $D \neq 0$, which implies that (2.6) has three distinct roots: $\lambda_0 = 1$, and λ_1, λ_2 either real or conjugate imaginary (with $\lambda_1 \lambda_2 = 1$).

Hence there are three distinct invariant points in V. For $\lambda = 1$ we have the point $Q_0(0, 0, 1)$ that is the point at infinity of s. For $\lambda = \lambda_1$ and $\lambda = \lambda_2$ we obtain the points Q_1 and Q_2 (distinct, real or imaginary) whose coordinates satisfy $z = 0$, the line l at infinity of the plane W parallel to m_1 and m_2. Summing up we have the following case

Ia. *U_1 and U_2 are not parallel; their line of intersection s is not parallel to W; U_1 is not parallel to m_2, U_2 is not parallel to m_1.*

There are no finite invariant points. Those in V are the vertices of a triangle $Q_0 Q_1 Q_2$; Q_0 is on s, Q_1, Q_2 are on W.

We suppose now $D = 0$, that is $A_2 B_1 = 0$, which implies $\lambda_1 = \lambda_2 = -1$. $\lambda = \lambda_0$ gives us once more the invariant point Q_0. If $A_2 \neq 0$, $B_1 = 0$ the only other invariant point is (0, 1, 0); for $A_2 = 0$, $B_1 \neq 0$ it is (1, 0. 0). Hence our case is

Ib. *U_1 and U_2 are not parallel; their intersection s is not parallel to W, U_1 is parallel to m_2, but U_2 is not parallel to m_1.* There are no finite invariant points. There are two invariant points in V, the point Q_0 and the point Q_{12}, the intersection of l and the plane U_1. We have a similar case if U_2 is parallel to m_1, but U_1 not parallel to m_2; the point Q_{12} is now the intersection of l and U_2.

If $A_2 = B_1 = 0$ it follows from (2.5) with $\lambda = -1$, that all points with $z = 0$ are solutions. Hence the case

Ic. *U_1 and U_2 are not parallel, s is not parallel to W, U_1 and m_2 are parallel and so are U_2 and m_1.* There are no finite invariant points, those in V are Q_0 and all points of l.

If U_1 and U_2 are parallel then s is not determined. $U(U_1, U_2)$ cannot be parallel to W (because U_i and m_i are not parallel). We take now OZ parallel to any line of U, not parallel to W. Completing the coordinate system as before implies that (2.1) and therefore (2.4) are still valid, but with the condition $A_2 B_1 = 1$. The equation (2.6) has then three equal roots $\lambda_0 = \lambda_1 = \lambda_2 = 1$. A solution of (2.5) is then any point satisfying $x + B_1 y = 0$ (or, what is the same thing, $A_2 x + y = 0$). Hence our next case is

Id. *U_1 and U_2 are parallel.* There are no finite invariant points. Those in V are all points on the intersection of U and V.

There is only one more case: the intersection s of U_1 and U_2 is parallel to W; the equations (2.1) do not hold. We take the coordinate system such that

$$(2.7) \qquad \begin{aligned} &m_1: \ y = 0, \ z = d; \qquad U_1: \ px + qy + r_1 z = 0, \\ &m_2: \ x = 0, \ z = -d; \quad U_2: \ px + qy + r_2 z = 0, \end{aligned}$$

with $r_1 \neq r_2$, $pq \neq 0$.

In an analogous way as before we obtain

$$R_1: x' = x + 2qp^{-1}y + 2r_1p^{-1}z - 2r_1p^{-1}d, \quad y' = -y, \quad z' = -z + 2d,$$
$$R_2: x' = -x, \quad y' = 2pq^{-1}x + y + 2r_2q^{-1}z + 2r_2q^{-1}d, \quad z' = -z - 2d,$$

and

$$R_2R_1: x' = -x - 2qp^{-1}y - 2r_1p^{-1}z + 2r_1p^{-1}d,$$
$$y' = 2pq^{-1}x + 3y + 2(2r_1 - r_2)q^{-1}z - 2(2r_1 - 3r_2)q^{-1}d, \tag{2.8}$$
$$z' = z - 4d.$$

As before there are no finite invariant points. The eigenvalues of the matrix of the linear terms of (2.8) are seen to be $\lambda_0 = \lambda_1 = \lambda_2 = 1$. Invariants points in V must satisfy the equations $px + qy + r_1z = 0$ and $px + qy + (2r_1 - r_2)z = 0$; as $r_1 \neq r_2$ their only solution is $(q, -p, 0)$. We have arrived at:

Ie. *U_1 and U_2 have the intersection s, parallel to W.* There are no finite invariant points; there is one in V: the point at infinity of s (which is on l).

We note that in all cases I if s exists, its point at infinity is an invariant point; this can even be said to hold in Id, where s is any line of U. A second remark: there is always at least one invariant point on l, the line at infinity of W.

3. The mirrors m_1, m_2 are intersecting lines. We can make use of the same coordinate systems as for case I, the only difference being that we have now $d = 0$. Hence $T = R_2R_1$ is again given by (2.4) and (2.8) respectively. We distinguish the same subcases as for I. As d does not appear in the linear terms our first conclusion is: for all subcases of II, the invariant points at infinity are the same as for the corresponding subcases of I. But there are now finite invariant points as well. They are the solutions of the equations $x' = x$, $y' = y$, $z' = z$. The results are

IIa. All points with $x = y = 0$, that means all points of s.

IIb. $A_2 \neq 0$, $B_1 = 0$; or $A_2 = 0$, $B_1 \neq 0$, once more the points of s.

IIc. $A_2 = B_1 = 0$: the points of s.

IId. $U_1 // U_2$: all points of the plane U, with equation $x + B_1y = 0$ (or $A_2x + y = 0$); that are all points of the (now undetermined) intersection s.

IIe. $s // U$: all points of $px + qy = z = 0$, that means all points of s.

Summing up we have: In every subcase the finite invariant points are those of s.

4. The mirrors m_1, m_2 are parallel lines. We take their plane W as $z = 0$, and OY parallel to m_1 and m_2. If U_1, U_2 have an intersection s, not parallel to W, we obtain for a suitable coordinate system:

$$m_1: x - a = z = 0; \quad U_1: A_1x + y = 0,$$
$$m_2: x + a = z = 0; \quad U_2: A_2x + y = 0, \tag{4.1}$$

with $a \neq 0$, $A_1 \neq A_2$.

We obtain

$$(4.2)\qquad \begin{aligned} R_1 &: x' = -x+2a, \quad y' = 2A_1x+y-2A_1a, \quad z' = -z,\\ R_2 &: x' = -x-2a, \quad y' = 2A_2x+y+2A_2a, \quad z' = -z, \end{aligned}$$

and therefore

$$(4.3)\qquad R_2R_1 : x' = x-4a, \quad y' = 2(A_1-A_2)x+y-2a(A_1-3A_2), \quad z' = z.$$

This implies that (in view of $a \neq 0$) there are no finite invariant points. For those in V we have $\lambda_0 = \lambda_1 = \lambda_2 = 1$ and it follows that all points of the plane $x=0$ are invariant. Hence

IIIa. *U_1 and U_2 have the line of intersection s, not parallel to W.* There are no finite invariant points; those in V are all points on its intersection with a plane parallel to s and m_i.

If U_1, U_2 are parallel we can again make use of (4.1), but now with $A_1 = A_2$. From (4.3) it follows:

IIIb. *The direction planes are parallel.* No finite invariant points. All points of V are invariant.

If the intersection s of U_1, U_2 is parallel to W a suitable coordinate system gives us

$$(4.4)\qquad \begin{aligned} m_1 &: x-a = z = 0; \quad U_1 : px+qy+r_1z = 0,\\ m_2 &: x+a = z = 0; \quad U_2 : px+qy+r_2z = 0, \end{aligned}$$

with $a \neq 0$, $r_1 \neq r_2$, $q \neq 0$.

After some algebra we obtain

$$(4.5)\qquad R_2R_1 : x' = x-4a, \quad y' = y+2(r_1-r_2)q^{-1}z+4pq^{-1}a, \quad z' = z.$$

This gives us the case:

IIIc. *The intersection s of U_1, U_2 is parallel to W.* There are no finite invariant points. Those in V satisfy $z=0$, they are the points at infinity of the plane W.

5. The mirrors m_1, m_2 coincide. The plane W is now undetermined, it can be any plane through m. We can still make use of (4.1) but now with $a=0$. If s exists it cannot be parallel to W, because U_1 and U_2 are not parallel to m. From (4.3), with $a=0$ it follows that there are only two subcases:

IVa. *U_1 and U_2 have the intersection s.* All points finite and infinite, of the plane $x=0$, that is the plane through s and m, are invariant.

IVb. *U_1 and U_2 are parallel.* All points of the space are invariant. We could expect this because $R_2 = R_1$ and $R_1^2 = I$.

6. Conclusion. In the preceding sections all possible pairs of affine line-reflections $R_i(m_i; U_i)$, $i = 1, 2$, have been considered and their product $T = R_2R_1$, an equi-affinity, has been discussed. For any T we determined the invariant points; invariant lines and planes could have been found by a similar procedure. The various transformations T belong to different types, but the most general equi-affinity (with the canonical representation $x' = k_1 x$, $y' = k_2 y$, $z' = k_3 z$, with distinct numbers k_i, satisfying $k_1 k_2 k_3 = 1$) is not among them. It has four invariant points, the vertices of a tetrahedron, with one finite vertex and three at infinity. To the negative conclusion that not all equi-affinities can be factorized in the way described, the lists of the sections 2, 3, 4, 5 add a survey of those types of (special) equi-affinities for which such a factorization is possible.

7. An algebraic method. An affinity is represented by

$$(7.1)\qquad \begin{aligned} x' &= a_{11}x + a_{12}y + a_{13}z + b_1, \\ y' &= a_{21}x + a_{22}y + a_{23}z + b_2, \\ z' &= a_{31}x + a_{32}y + a_{33}z + b_3, \end{aligned}$$

and it is an equi-affinity if $|a_{ij}| = 1$ We can extend (7.1) to a projectivity if we introduce homogeneous coordinates and add the line $w' = w$. Two matrices correspond to (7.1): the 4×4 matrix M_1 of the projective transformation and the 3×3 matrix $M_2 = \| a_{ij} \|$, which expresses the projective transformation in the plane V at infinity. It is immediately seen that $\lambda = 1$ is an eigenvalue of M_1, and moreover that the eigenvalues of M_1 are $\lambda_4 = 1$ and the eigenvalues $\lambda_0, \lambda_1, \lambda_2$ of M_2. But (2.4) and (2.8) show that for a suitably chosen coordinate system we have for the product $R_2 R_1 : a_{31} = a_{32} = 0$, $a_{33} = 1$, which implies that one eigenvalue of M_2 is $\lambda_0 = 1$. *Hence a product* $R_2 R_1$, *considered as a projective transformation of the space, has* (at least) *two equal eigenvalues and therefore cannot be one of the general type.*

Two projectivities are equivalent if their matrices M and M' are equivalent, which means that a matrix A exists such that $M' = A^{-1}MA$. For the equivalence of M and M' it is not only necessary that they have the same set of eigenvalues, but moreover that their *elementary divisors* are the same, which results in λ_k being a common (multiple) eigenvalue, the matrices $M - \lambda_k I$ and $M' - \lambda_k I$ having the same rank. This theory leads to an algebraic classification of projectivities at which every type is characterized by its so-called symbol of SEGRE. For the sake of completeness we give a list of these symbols, for M_1 and M_2, corresponding to the various types of equi-affinities discussed in the preceding sections.

	M_1	M_2		M_1	M_2
Ia.	[2 1 1],	[1 1 1]	IIa.	[(11) 1 1],	[1 1 1]
b.	[2 2],	[2 1]	b.	[2 (11)],	[2 1]
c.	[2 (11)],	[1 (11)]	c.	[(11) (11)],	[1 (11)]
d.	[(2 2)],	[(2 1)]	d.	[(2 1 1)],	[(2 1)]
e.	[4],	[3]	e.	[(3 1)],	[3]
IIIa.	[(3 1)],	[(2 1)]	IVa.	[(2 1 1)],	[(2 1)]
b.	[(2 1 1)],	[(1 1 1)]	b.	[[1 1 1 1)],	[(1 1 1)].

We remark that all (six) types of projective transformations in V appear in our scheme. But those in the space as a whole are restricted to ten out of the fourteen types which are possible. The equi-affinities of the types [1 1 1 1], [3 1], [(2 1) 1] and [(1 1 1) 1] cannot be written as the product of two affine line-reflections.

1. Saly Ruth Ramler: *Axiomatik der affinen Geometrie in zwei und drei Dimensionen.* Dissertation Karl Universität, Prag, 1919.

In consequence of the war the thesis could not be printed. By Mrs. Struik's courtesy I have been able to read an english translation; for a survey see:

Saly Ruth Struik: *A theorem on equiareal triangles.* Supplement to For Dirk Struik (Dordrecht/Boston, 1974).

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