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586. TRANSFORMATION OF THE CONTINUED FRACTION INTO A RATIONAL FUNCTION*

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This paper presents the transfortmation of the coefficients of the continued fraction into those of the rational function. The algorithm is represented by formulas and by the fortran program.

The continued fractions of some functions have been known for several centuries, but their use was relatively nonsignificant. The advantage of the continued fractions over the potential ones are certain: the domain of application of continued fractions is much larger and the number of operations to achieve the prescribed accuracy is considerably smaller. The trouble, due to which the continued fractions have not been used to a sufficient extent, is the need of a large number of divisions — so that it is natural that without the computer they could not be used in a larger extent. However, numerous results have been achieved. See References.

By an increasing use of the computers, the continued franctions are becoming more and more important. The situation changes due to the fact that the computer performs the divisions. But, even for the computer, the division, especially of the complex numbers, is a relatively lengthy operation. That is why one endeavours to reduce a series of divisions into a single division and a much quicker operation — multiplication, which is done in this paper.

Staring from coefficients $A_k\ (k=1\,(1)\,n)$ of the continued fraction:

$$f(z)=\frac{A_1}{z+}\,\frac{A_2}{z+}\,\frac{A_3}{z+}\cdots\frac{A_{n-1}}{z+}\,\frac{A_n}{z}$$

coefficients $R_{k,n}\ (k=1\,(1)\,n+1)$ of the rational function

$$f(z)=\left(\frac{R_{2,n}\,z+R_{4,n}\,z^3+R_{4,n}\,z^5+\cdots}{R_{1,n}+R_{3,n}\,z^2+R_{5,n}\,z^4+\cdots}\right)^{(-1)^n}$$

can be calculated by means of the following algorithm:

$$R_{1,1}=A_n;$$

$$R_{j+1,j}=1 \quad (j=1\,(1)\,n);$$

$$R_{k,j}=\begin{cases} R_{k,j-1}\,A_{n+1-j} & (k+j \text{ even}) \\ R_{k,j-1}+R_{k-1,\,j-1} & (k+j \text{ odd},\ k>1), \\ R_{1,j-1} & (j \text{ even}) \quad (k=j\,(-1)\,1,\ j=2\,(1)\,n). \end{cases}$$

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For the real argument — the value of the continued fraction is obtained by n divisions and by $n-1$ additions, while for $n>1$ the value of the rational function is obtained by one division, n multiplications and $n-1$ additions. For the complex argument the savings are much greater.

The proof of the algorithm is based on induction, see for example [**12**]. For $n=1$ the continued fraction becomes:

$$f(z)=\frac{A_1}{z},$$

so that $R_{1,1}=A_1$, $R_{2,1}=1$ – which means that the algorithm is valid then. Since from the assumption that the statement is valid for the natural number $j-1$ we prove that it is valid for the next natural number j, we may conclude that is valid for any natural number n.

$$f(z)=\frac{A_1}{z+}\cdots\frac{A_{n-3}}{z+}\,\frac{A_{n-2}}{z+}\,\frac{A_{n-1}}{z+}\,\frac{A_n}{z}$$

$$=\frac{A_1}{z+}\cdots\frac{A_{n-3}}{z+}\,\frac{A_{n-2}}{z+}\,\frac{A_{n-1}\,z}{A_n+z^2}$$

$$=\frac{A_1}{z+}\cdots\frac{A_{n-3}}{z+}\,\frac{A_n A_{n-2}+A_{n-2}\,z^2}{(A_n+A_{n-1})\,z+z^3}$$

$$\vdots$$

$$=\frac{A_1}{z+}\cdots\frac{A_{n-j}}{z+}\,\frac{A_{n+1-j}}{z+}\left(\frac{R_{2,j-1}\,z+R_{4,j-1}\,z^3+\cdots}{R_{1,j-1}+R_{3,j-1}\,z^2+\cdots}\right)^{(-1)^{j-1}}$$

$$=\begin{cases}\dfrac{A_1}{z+}\cdots\dfrac{A_{n-j}}{z+}\left(\dfrac{(R_{1,j-1}+R_{2,j-1})\,z+(R_{3,j-1}+R_{4,j-1})\,z^3+\cdots+z^j}{R_{1,j-1}A_{n+1-j}+R_{3,j-1}\,A_{n+1-j}\,z^2+\cdots}\right)^{(-1)^j} & (j \text{ odd})\\[2ex] \dfrac{A_1}{z+}\cdots\dfrac{A_{n-j}}{z+}\left(\dfrac{R_{2,j-1}\,A_{n+1-j}\,z+R_{4,j-1}\,A_{n+1-j}\,z^3+\cdots}{R_{1,j-1}+(R_{2,j-1}+R_{3,j-1})\,z^2+\cdots+z^j}\right)^{(-1)^j} & (j \text{ even})\end{cases}$$

$$=\frac{A_1}{z+}\cdots\frac{A_{n-j}}{z+}\left(\frac{R_{2,j}\,z+R_{4,j}\,z^3+R_{6,j}\,z^5+\cdots}{R_{1,j}+R_{3,j}\,z^2+R_{5,j}\,z^4+\cdots}\right)^{(-1)^j}$$

$$\vdots$$

$$=\left(\frac{R_{2,n}\,z+R_{4,n}\,z^3+R_{6,n}\,z^5+\cdots}{R_{1,n}+R_{3,n}\,z^2+R_{5,n}\,z^4+\cdots}\right)^{(-1)^n}.$$

Since matrix R would take much of the memory space (its dimension being $(n+1)\times n$), it is substituted by the vector of dimension $(n+1)$ in the fortran program, so that $R(k)=R_{k,j}$. The second indices are superfluous if j

increases from 1 to n, and k decreases from $j+1$ to 1, see the table of values $R_{k,j}$. The algorithm becomes:

$$R_k = 1 \quad (k = 1\,(1)\,n+1)$$

$$R_k = \begin{cases} R_k\, A_{n+1-j} & (k+j \text{ even}) \\ R_k + R_{k-1} & (k+j \text{ odd},\ k>1) \end{cases} \qquad ((k=j\,(-1)\,1),\ j=1\,(1)\,n).$$

The given algorithm is distinguished by its simplicity. Compare for example [6], p. 155 — 160. Thus — in that way the algorithm bacomes usable even for pocket computers. On the basis of this algorithm, the VNRN subroutine was written in the basic fortran, to be used on computers of various types.

The table of values of $R_{k,j}$

j \ k	1	2	3	4	5
1	A_n	1	0	0	0
2	A_n	A_{n-1}	1	0	0
3	$A_n A_{n-2}$	$A_n + A_{n-1}$	A_{n-2}	1	0
4	$A_n A_{n-2}$	$(A_n + A_{n-1}) A_{n-3}$	$A_n + A_{n-1} + A_{n-2}$	A_{n-3}	1

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      SUBROUTINE VNRN (N,A,R)
C     TRANSFORMATION OF THE COEFFICIENTS A(K) OF THE CONTINUED
C     FRACTION  A(1)/(Z+A(2)/(Z+...A(N-1)/(Z+A(N)/Z)...))
C     INTO THE COEFFICIENTS R(K) OF THE RATIONAL FUNCTION
C     ((R(2)*Z+R(4)*Z**3+...)/(R(1)+R(3)*Z**2+...))**((-1)**N).
C     N - INPUT ORDER OF THE CONTINUED FRACTION.
C     A - INPUT VECTOR OF COEFFICIENTS OF THE CONTINUED
C         FRACTION. WITH DIMENSION  N.
C     B - OUTPUT VECTOR OF COEFFICIENTS OF THE RATIONAL
C         FUNCTION. WITH DIMENSION  N+1.
      DIMENSION A(1),R(1)
      L=N
      R(1)=1
    1 K=N+2-L
      R(K)=1
      GO TO 3
    2 R(K)=R(K)+R(I)
    3 K=K-1
      R(K)=R(K)*A(L)
      K=K-1
      I=K-1
      IF(I) 4,4,2
    4 L=L-1
      IF(L) 5,5,1
    5 RETURN
      END
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D. Blanuša, D. S. Mitrinović, B. Crstici, and A. Žepić have read this paper in manuscript and have made some valuable remarks and sugestions.

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