UNIV. BEOGRAD PUBL. ELEKTROTHEN. FAK.
Ser. Mat. Fiz. № 577 — № 598 (1977), 45—48.

585. THEOREMS ON NONLINEAR SUPERPOSITIONS: THE GENERAL FIRST ORDER EQUATION*

Vlajko Lj. Kocić

0. Denote by $S(E)$ the set of all solutions of partial differential equation (E). If

$$u,\ v \in S(E) \quad \text{implies} \quad F(u,\ v) \in S(E),$$

the function F is called a „connecting function" for (E). It defines a nonlinear superposition for (E).

In paper [1] J. D. KEČKIĆ found nonlinear superpositions for the equations

(0.1) $$aU_x + bU_y + cU = 0,$$

(0.2) $$aU_x + bU_y + cU + d = 0,$$

(0.3) $$aU_x + bU_y + c\varphi(U) = 0,$$

where a, b, c, d are functions of x, y and φ is a given function. His result reads:

For equations (0.1), (0.2), (0.3) nonlinear superpositions are given by $F(u, v) = uf\left(\frac{v}{u}\right)$, $F(u, v) = u + \alpha(v - u)$, $\int \frac{dF}{\varphi(F)} = \int \frac{du}{\varphi(u)} + f\left(\int \frac{dv}{\varphi(v)} - \int \frac{du}{\varphi(u)}\right)$ respectively, where α is an arbitrary constant and f is an arbitrary function.

In this paper we consider the more general equation

(E_0) $$U_x = \Phi(U_y, U, x, y),$$

where Φ is a twice differentiable function, and give a set of sufficient and necessary conditions such that the implication

(I) $$u,\ v \in S(E_0) \quad \text{implies} \quad F(u,\ v) \in S(E_0)$$

is valid, where F is a differentiable function.

REMARK. We consider equation (E_0) instead of the general equation

(0.4) $$\Phi(U_x, U_y, U, x, y) = 0$$

for technical reasons. Similar conclusions can be obtained for equation (0.4).

1. Theorem. *A differentiable function $F = F(u, v)$ is a connecting function for (E_0) if and only if:*

(*i*) *equation (E_0) is of the form*

(E_1) $$g_U(U, x, y)(aU_x + bU_y) + cg(U, x, y) = 0$$

where a, b, c are functions of x, y and g is a differentiable function;

* Presented April 5, 1977 by D. S. MITRINOVIĆ and J. D. KEČKIĆ.

(*ii*) *there exist a differentiable function f such that*:

$$g(F, x, y) = g(u, x, y) f\left(\frac{g(v, x, y)}{g(u, x, y)}\right), \tag{1.1}$$

$$g_x(F, x, y) = g_x(u, x, y) f\left(\frac{g(v, x, y)}{g(u, x, y)}\right) \tag{1.2}$$
$$+ f'\left(\frac{g(v, x, y)}{g(u' x, v)}\right)\left(g_x(v, x, y) - g_x(u, x, y)\frac{g(v, x, y)}{g(u, x, y)}\right),$$

$$g_y(F, x, y) = g_y(u, x, y) f\left(\frac{g(v, x, y)}{g(u, x, y)}\right) \tag{1.3}$$
$$+ f'\left(\frac{g(v, x, y)}{g(u, x, y)}\right)\left(g_y(v, x, y) - g_y(u, x, y)\frac{g(v, x, y)}{g(u, x, y)}\right).$$

Proof. 1° Sufficient conditions. Let $u, v \in S(E_1)$.

Differentiating (1.1) with respect to x and y and using (1.2) and (1.3), we find

$$g_F(F, x, y)\,(F_u u_x + F_v v_x) = g_u(u, x, y)\, u_x f \tag{1.4}$$
$$+ f' \cdot \left(g_v(v, x, y)\, v_x - g_u(u, x, y)\, u_x \frac{g(v, x, y)}{g(u, x, y)}\right)$$

and

$$g_F(F, x, y)\,(F_u u_y + F_v v_y) = g_u(u, x, y)\, u_y f \tag{1.5}$$
$$+ f' \cdot \left(g_v(v, x, y)\, v_y - g_u(u, x, y)\, u_y \frac{g(v, x, y)}{g(u, x, y)}\right).$$

Using (1.1), (1.4) and (1.5), we get that $F = F(u, v)$ is also a solution of equation (E_1), i.e. F is a connecting function for (E_1), which proves the first part of the theorem.

2° Necessary conditions. Let $u, v \in S(E_0)$ and (I) holds, i.e. let $F(u, v) \in S(E_0)$. Then we have $u_x = \Phi(u_y, u, x, y)$, $v_x = \Phi(v_y, v, x, y)$, $F_x = F_u u_x + F_v v_x = \Phi(F_y, F, x, y)$ and

$$F_u \Phi(u_y, u, x, y) + F_v \Phi(v_y, v, x, y) = \Phi(F_u u_y + F_v v_y, F, x, y). \tag{1.6}$$

Differentiating (1.6) with respect to u_y and v_y we get

$$\frac{\partial^2 \Phi(F_y, F, x, y)}{\partial F_y^2} = 0. \tag{1.7}$$

Hence

$$\Phi(t_1, t_2, x, y) = A(t_2, x, y)\, t_1 + B(t_2, x, y), \tag{1.8}$$

where A and B are arbitrary functions of t_2, x, y.

Substituting (1.8) into (1.6), we find

$$A(u, x, y) = A(v, x, y) = A(F, x, y) = -\frac{b(x, y)}{a(x, y)}, \tag{1.9}$$

where a and b are arbitrary functions of x and y.

Also we have

$$F_u B(u, x, y) + F_v B(v, x, y) = B(F, x, y). \tag{1.10}$$

Putting

$$B(t, x, y) = -\frac{c(x, y)}{a(x, y)} \frac{g(t, x, y)}{g_t(t, x, y)} \tag{1.11}$$

into (1.10), we get

$$\frac{g(u, x, y)}{g_u(u, x, y)} F_u + \frac{g(v, x, y)}{g_v(v, x, y)} F_v = \frac{g(F, x, y)}{g_F(F, x, y)}. \tag{1.12}$$

Also, from (1.8) and (1.11) we obtain that equation (E_0) is of the form (E_1), i.e. the condition (i) is fulfiled.

Furthermore, the general solution of equation (1.12) is given by

$$g(F, x, y) = g(u, x, y) f\left(\frac{g(v, x, y)}{g(u, x, y)}\right), \tag{1.13}$$

where f is an arbitrary differentiable function.

Differentiating (1.13) with recpect to x and y, we get

$$\begin{aligned} g_F(F, x, y) F_x + g_x(F, x, y) &= \big(g_u(u, x, y) u_x + g_x(u, x, y)\big) f \\ &+ f' \cdot \big(g_v(v, x, y) v_x + g_x(v, x, y)\big) - \frac{g(v, x, y)}{g(u, x, y)} \big(g_u(u, x, y) u_x + g_x(u, x, y)\big) \end{aligned} \tag{1.14}$$

and

$$\begin{aligned} g_F(F, x, y) F_y + g_y(F, x, y) &= \big(g_u(u, x, y) u_y + g_y(u, x, y)\big) f \\ &+ f' \cdot \big(g_v(v, x, y) v_y + g_y(v, x, y)\big) - \frac{g(v, x, y)}{g(u, x, y)} \big(g_u(u, x, y) u_y + g_y(u, x, y)\big). \end{aligned} \tag{1.15}$$

Since $F(u, v) \in S(E_1)$ we have $g_F(F, x, y)(aF_x + bF_y) + g(F, x, y) = 0$.

From the above and (1.13), (1.14), (1.15) we obtain (1.2) and (1.3).

All connecting functions for equation (E_1) may be determined from (1.13), (1.2), (1.3). This means that $F = F(u, v)$ is a connecting function for (E_1) if and only if there exists a function f such that (1.1), (1.2), (1.3) are valid. This completes the proof.

2. Remarks. 1° Theorem can be formulated in the following way:

The only first order partial differential equation which has a connecting function of the form $F(u, v)$ is of the form (E_1).

$F = F(u, v)$ defines a nonlinear superposition for (E_1) if and only if there exists differentiable function f such that (1.1), (1.2) and (1.3) holds.

2° Let $g(t, x, y)=t+d/c$ (d is a function of x, y; d/c is not a constant). Then (E_1) becomes

$$aU_x+bU_y+cU+d=0. \tag{2.1}$$

In this case (1.1) takes the form

$$F(u, v)+\frac{d}{c}=\left(u+\frac{d}{c}\right) f\left(\frac{v+(d/c)}{u+(d/c)}\right) \tag{2.2}$$

and from (1.2) and (1.3) it follows

$$1=f(t)+f'(t)(1-t), \tag{2.3}$$

where $t=\dfrac{v+(d/c)}{u+(d/c)}$.

The general solution for equation (2.3) is $f(t)=\alpha(t-1)+1$ where α is an arbitrary constant.

Thus the nonlinear superposition for equation (2.1) is defined by $F(u, v)=u+\alpha(v-u)$. This result was also obtained by J. D. Kečkić in paper [1].

3° In the case when $g(t, x, y)=g(t)$ we have the equation

$$g'(U)(aU_x+bU_y)+cg(U)=0. \tag{2.4}$$

For this equation nonlinear superposition is defined by

$$g(F(u, v))=g(u) f\left(\frac{g(v)}{g(u)}\right),$$

where f is an arbitrary function.

Equation (0.3) reduces to equation (2.4) after the substitution $\varphi(U)=\dfrac{g(U)}{g'(U)}$, and our result reduces to result of J. D. Kečkić, for the equation (0.3).

4° Let $g(t, x, y)=g(t)+d/c$ (d is a function of x and y such that d/c is not a constant). Then we have the equation

$$g'(U)(aU_x+bU_y)+cg(U)+d=0$$

for which the nonlinear superposition is defined by

$$g(F(u, v))=g(u)+\alpha(g(v)-g(u)),$$

where α is an arbitrary constant.

3. We shall return to this topic and shall, in particular, investigate second order equations in an other paper.

REFERENCE

1. J. D. Kečkić: *On nonlinear superposition.* Math. Balkanica **2** (1972), 88—93.