UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat Fiz. № 577—№ 598 (1977), 39—44.

584. MULTIPLE TRIANGLE INEQUALITIES*

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1. Introduction. Let P and P' be interior points of triangle $A_i\,(i=1,\ 2,\ 3)$ with sides a_i, circumradius R and area Δ. If $R_i=PA_i$, $R_i'=P'A_i$, I give, first, a proof that

$$(1) \qquad a_1 R_1 R_1' + a_2 R_2 R_2' + a_3 R_3 R_3' \geqq 4 R \Delta$$

with equality if and only if P and P' are isogonal conjugates.

Next, isogonal, inversion and reciprocation transformations on (1) will yield dual inequalities along with some special cases. (KLAMKIN **[1]** and OPPENHEIM **[2]**)

The main part of this artcle consists of a new proof of the two-triangle inequality of BOTTEMA [**3**, 12.56]

$$(2) \qquad (a_1 x + b_1 y + c_1 z)^2 \geqq (M/2) + 8\,\Delta\Delta'$$

where $M=\sum a_1^2 (b^2+c^2-a^2)$; a_1, b_1, c_1 and a, b, c are sides of two arbitrary triangles of area Δ', Δ respectively and x, y, z are the distances from an interior point P of triangle ABC to the respective vertices. (1) will play an important role in my proof of (2) and also cast new light on the conditions for equality. (BOTTEMA, KLAMKIN **[4]**)

Finally, my proof of (2) will, in turn, suggest a new three-triangle inequality. The article concludes with some special cases and general observations.

2. ***Proof of*** **(1).** Using figure A as an aid, assume given triangle A_i with sides a_i and area Δ, and let triangle B_i be the pedal triangle of the interior point P with sides b_i. If P' is also any interior point, let $R_i=PA_i$, $R_i'=P'A_i$, and R_i' intersect b_i at angle θ_i. Draw $P'B_1$, $P'B_2$ $P'B_3$ partitioning triangle A_i

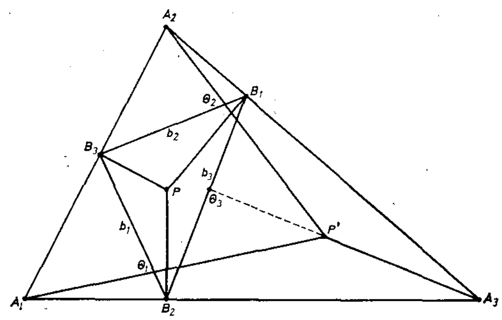

Figure A

* Presented March 15, 1977 by O. BOTTEMA and R. R. JANIĆ.

into quadrilaterals $P'B_2A_1B_3$, $P'B_3A_2B_1$ and $P'B_1A_3B_2$ whose areas are respectively $(1/2)\,b_i R_i' \sin\theta_i$. Since $\sin\theta_i \leqq 1$, $\sum b_i R_i' \geqq 2\Delta$. Now, (COURT [5] and JOHNSON [8]) $\theta_1 = \theta_2 = \theta_3 = 90°$ if and only if P' is the isogonal conjugate of P. Since $b_i = a_i R_i/2R$, this concludes the proof of (1).

3. Special Cases of (1). If P' = circumcenter (O), (1) reduces to

$$\sum R_i a_i \geqq 4\Delta$$

with equality if and only if P = orthocenter (H). This inequality can be used to give an alternate proof to the classic theorem: in acute triangles, the inscribed triangle having minimal perimeter is the orthic triangle. (PEDOE [6], KAY [7])

For another special case, let $P = P'$. Then, (1) simplifies to

$$\sum R_i^2 a_i \geqq 4R\Delta \tag{3}$$

with equality when P = incenter (I).

An inversion transformation on (3) yields a dual inequality

$$\sum a_1 R_2 R_3 \geqq 4R\Delta \tag{4}$$

with equality when $P = H$.

A reciprocation transformation on (3) yields

$$\sum a_1 R_1 r_2 r_3 \geqq \frac{\Delta R_1 R_2 R_3}{R} \tag{5}$$

with equality when $P = O$.

As isogonal transformation on (3) yields

$$\sum a_1 (r_1 R_1)^2 \geqq \frac{\Delta (R^2 - PO^2)^2}{R} \tag{6}$$

with equality when $P = I$. Multiple triangle inequalities can easily be derived using (3)—(6) by letting P vary over the interior of triangle A_i. For example, if $P = I$, (5) reduces to $\sum \cos A_i/2 \geqq \sum \sin A_i$. This result can be derived directly as follows:

$$\begin{aligned}\frac{1}{2}(\sin A_1 + \sin A_2) &= \sin(A_1 + A_2)/2 \cos(A_1 - A_2)/2 \\ &= \cos A_3/2 \cos(A_1 - A_2)/2 \\ &\leqq \cos A_3/2.\end{aligned}$$

Thus, $\sum \sin A_i \leqq \sum \cos A_i/2$ with equality if triangle A_i is equilateral.

Finally, let triangle A_i be equilateral with side a in (1). If P' is at the center, (1) reduces to

$$R_1 + R_2 + R_3 \geqq \sqrt{3}\, a \tag{7}$$

and one way this result can be verified independently is to show

$$\sum R_1 \sin(60° + \theta_1) = \sqrt{3}\, a$$

where θ_1 = angle A_3A_1P, etc.

4. ***Proof of*** **(2).** Changing to a more convenient notation, assume given two arbitrary triangles A_i and A_i' with sides a_i and a_i' respectively. Let $R_1 = PA_1$, etc. where P is any interior point of triangle A_i, and

$$\Phi = \sum a_1'^2 (a_2^2 + a_3^2 - a_1^2) + 16\Delta\Delta'.$$

BOTTEMA's inequality now takes on the following form:

$$(\sum a_1' R_1)^2 \geqq \frac{\Phi}{2}. \tag{8}$$

To prove (8), construct outwordly on the sides of triangle A_i, triangles similar to triangle A_i' in the following sense: (see figure B)

$$X_1 A_3 A_2 \sim A_1 A_3 X_2 \sim A_1 X_3 A_2.$$

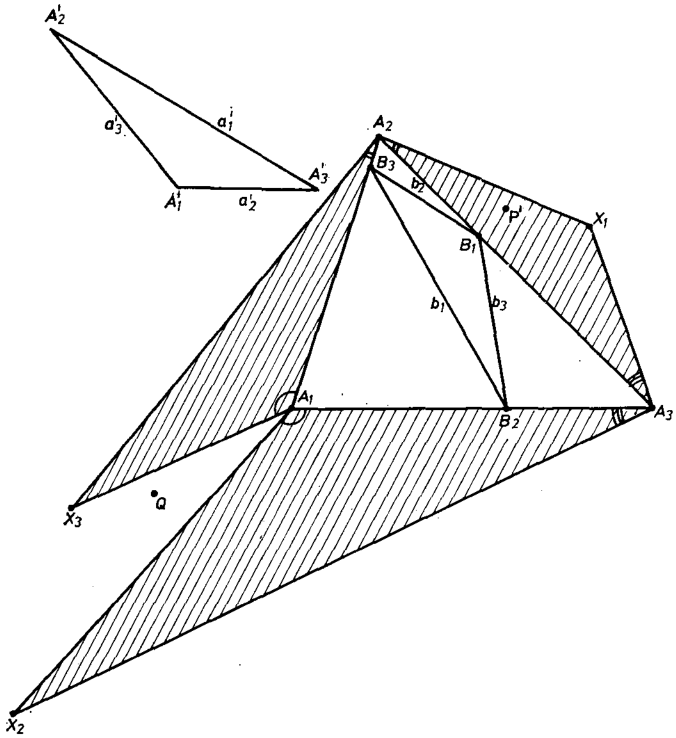

Figure B

It is known (JOHNSON [8]) that A_1X_1, A_2X_2, and A_3X_3 intersect at a point. Call this point Q. If P' is the isogonal conjugate of Q, the pedal triangle B_i of P' is similiar to triangle A_i'. Let b_i be the sides of triangle B_i, $R_i' = P'A_i$ and $r_i' = P'B_i$. It is now easy to show that

$$a_1' = \frac{1}{2\Delta}\sqrt{\frac{\Phi}{2}}\, b_1, \text{ etc., } R_1' = \sqrt{\frac{2}{\Phi}}\, a_1' a_2 a_3, \text{ etc., } r_1' = \frac{8R\Delta}{\Phi}\, a_2' a_3' \sin(A_1 + A_1'), \text{ etc.}$$

(In passing, we note the above construction gives one possible solution to the problem—given any two triangles, to inscribe a triangle similar to one triangle inside the other triangle.) Now, let P be any interior point in triangle A_i, and $R_1 = PA_1$, etc. Clearly, $a_1'R_1 = \frac{1}{2\Delta}\sqrt{\frac{\Phi}{2}}\, b_1 R_1$ and summing over the three sides yields

$$(9) \qquad (\sum a_1'R_1)^2 = \frac{\Phi}{2}\left[\frac{\sum b_1 R_1}{2\Delta}\right]^2.$$

Since it has been shown above that $\sum b_1 R_1 \geqq 2\Delta$ and from figure B, $b_1 = a_1 R_1'/2R$, I conclude

$$(\sum a_1'R_1)^2 = \frac{\Phi}{2}\left[\frac{\sum a_1 R_1 R_1'}{4R\Delta}\right]^2 \geqq \frac{\Phi}{2}$$

with equality only if $\sum a_1 R_1 R_1' = 4R\Delta$ and that can happen if and only if P and P' are isogonal conjugates.

5. Special Cases of (8). If triangle A_i and A_i' are similiar, Φ is proportional to $32\Delta^2$ and (8) reduces to $\sum a_1 R_1 \geqq 4\Delta$, a special case derived earlier.

Another special case is to leave unchanged the two triangles while applying within triangle A_i an isogonal transformation. Then (8) reduces to the following:

$$(10) \qquad (\sum a_1' r_1 R_1)^2 \geqq \frac{\Phi(\sum a_1 r_2 r_3)^2}{8\Delta^2}.$$

If the two triangles are similar (10) reduces to

$$(11) \qquad \sum a_1 r_1 R_1 \geqq 2\sum a_1 r_2 r_3$$

with equality when $P = O$.

Another special case is to let $P = P'$. If $R_1 = \sqrt{\frac{2}{\Phi}}\, a_1' a_2 a_3$, (8) reduces to

$$(12) \qquad \sum a_1'^2 a_2 a_3 \geqq \frac{\Phi}{2}$$

and again letting the two triangles be similiar reduces (12) to Euler's well-known inequality $R \geqq 2r$. If, instead, triangle A_i' is similiar to the reciprocal triangle relative to P, then (12) transforms to

$$(13) \qquad \sum a_1 (r_1 R_1)^2 \geqq \frac{\Delta (R^2 - PO^2)^2}{R}$$

with equality when $P = I$.

6. A Three-triangle Inequality. Our starting point is the known inequality (Klamkin [9])

$$(14) \qquad (\sum w_1)(\sum w_1 R_1^2) \geqq \sum a_1^2 w_2 w_3$$

where P is any interior point of triangle A_i with sides a_i, $R_1 = PA_1$, etc. and w_i are real numbers. There is equality *iff* $a_1 r_1/w_1 = a_2 r_2/w_2 = a_3 r_3/w_3$. Let

$$R_1 = = \sqrt{\frac{2}{\Phi}}\, a_1' a_2 a_3$$

in (14). Then,

$$\sum w_1 a_1'^2 a_2^2 a_3^2 \geqq \frac{\Phi \sum a_1^2 w_2 w_3}{2 \sum w_1} \tag{15}$$

with equality *iff*

$$a_1 a_2' a_3' \sin(A_1 + A_1')/w_1 = a_2 a_3' a_1' \sin\ A_2 + A_2')/w_2 = a_3 a_1' a_2' \sin(A_3 + A_3')/w_3.$$

(15) can be viewed as a three-triangle inequality by restricting the w_i to be sides of a third arbitrary triangle W_i. Letting any combination of the three triangles take on special restrictions and/or placing restrictions on the interior point P of triangle A_i clearly yields a large number of triangle inequalities and equalities. Inversion and reciprocation transformations yield dual inequalities often difficult to show by more elementary methods.

7. Special Cases of (15). If triangle A_i' is similiar to triangle A_i, then Φ is proportional to $32\,\Delta^2$ and (15) reduces to the following:

$$R^2 (\sum w_i)^2 \geqq \sum a_1^2 w_2 w_3. \tag{16}$$

If w_i form the sides of a triangle W_i, there will be equality if and only if triangle W_i is similar to the orthic triangle of acute triangle A_i. (This is *Problem E* 2221, Amer. Math. Monthly **78** (1971), 82 — 83).

Again, let triangle A_i' be similiar to the reciprocation triangle relative to P in triangle A_i. (15) reduces to

$$\sum w_1 (r_1 R_1)^2 \geqq \frac{(R^2 - PO^2)^2 \sum a_1^2 w_2 w_3}{4R^2 \sum w_1} \tag{17}$$

with equality if and only if $r_1 w_1/a_1 = r_2 w_2/a_2 = r_3 w_3/a_3$. If triangle W_i is similiar to triangle A_i, (17) reduces to (6).

REMARK. An isogonal transformation on (14) yields (17) also. Indeed, the reciprocation transformation relative to P, while clever and convenient, is not really new—it is just an inversion transformation with respect to the isogonal conjugate of P.

Finally, if triangle A_i' is similiar to the inversion triangle of P in triangle A_i, (15) reduces to (14). Thus, (15) is, *indeed*, a more general triangle inequality.

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