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580. ON $\binom{k^2}{k}$ AND THE PRODUCT OF THE FIRST k PRIMES*

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1. Introduction. We observe that

$$\binom{1^2}{1}=1<2,\quad \binom{2^2}{2}=6=2\cdot 3,\quad \binom{3^2}{3}=84>2\cdot 3\cdot 5,$$

$$\binom{4^2}{4}=1820>2\cdot 3\cdot 5\cdot 7,\quad \binom{5^2}{5}=53130>2\cdot 3\cdot 5\cdot 7\cdot 11.$$

From these examples, it will appear that for any positive integer $k>2$, $\binom{k^2}{k}$ is greater than the product of the first k primes

The object of this note is to show that this is true only for $2<k<1794$, while for $k\geqq 1794$, $\binom{k^2}{k}$ is less than the product of the first k primes.

Our method of establishing the result is simple and straightforward. Let

$$A(k)=\ln\binom{k^2}{k}\quad\text{and}\quad P(k)=\sum_{j=1}^{k}\ln p_j$$

where p_j denotes the jth prime (here and in what folows).
Then all we need show is that

$$P(k)-A(k)<0\quad\text{for}\quad 2<k<1794$$
$$>0\quad\text{for}\quad k\geqq 1794.$$

For this purpose, the first author proceeded to obtain a formula for $A(k)$, while the second author computed a table of values of $P(k)$ for values of $k\leq 2262$ (covering all primes below 20000). The table was rechecked by the two authors together and we are certain there are no mistakes. We made a free use of the "Tables of natural logarithms" and the "Handbook of mathematical functions" published by the National Bureau of Standards, U. S. Department of Commerce, Washington, D. C.

Professor D. H. LEHMER kindly used his computer at Berkeley and confirmed our results.

A graph of the function $P(k)$—$A(k)$ is presented at the end of the paper. It shows that $P(k)$—$A(k)$ is least at $k=617$. This implies that for each natural number k, $\binom{k^2}{k}$ is less than the product of the first $k+13$ primes.

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2. A formula for $\ln\binom{k^2}{k}$**.** We have $\binom{k^2}{k}=N/D$, where

$$N=k^2(k^2-1)(k^2-2)\cdots(k^2-k+1), \qquad D=k!.$$

Evidently

$$\ln N=2k\ln k+\sum_{j=1}^{k-1}\ln\left(1-\frac{j}{k^2}\right) \tag{2.1}$$

$$=2k\ln k-\sum_{j=1}^{k-1}\left\{\frac{j}{k^2}+\frac{j^2}{2k^4}+\frac{j^3}{3k^6}+\cdots\right\}$$

$$=2k\ln k-0.5+(3k)^{-1}+O(k^{-2}).$$

On the other hand, we have

$$\ln D=\left(k+\frac{1}{2}\right)\ln k+\sum_{j=1}^{k-1}\left(j+\frac{1}{2}\right)\ln\left(\frac{j}{j+1}\right) \tag{2.2}$$

$$=\left(k+\frac{1}{2}\right)\ln k+\sum_{j=1}^{k-1}\left(j+\frac{1}{2}\right)\ln\left(1-\frac{1}{j+1}\right)$$

$$=\left(k+\frac{1}{2}\right)\ln k+\sum_{j=2}^{k}\left(j-\frac{1}{2}\right)\ln\left(1-\frac{1}{j}\right)$$

$$=\left(k+\frac{1}{2}\right)\ln k-\sum_{j=2}^{k}\left(j-\frac{1}{2}\right)\left(\frac{1}{j}+\frac{1}{2j^2}+\frac{1}{3j^3}+\cdots\right).$$

Now, the expression on the right of the sigma in (2.2)

$$=1+\left\{\frac{1}{3\cdot 4}j^{-2}+\frac{2}{4\cdot 6}j^{-3}+\cdots+\frac{(r-1)}{(r+1)\cdot 2r}j^{-r}+\cdots\right\}$$

$$=1+f(j), \text{ say.}$$

Evidently, then

$$\sum_{j=2}^{k}1=k-1. \tag{2.3}$$

Moreover

$$12f(j)<(j^{-2}+j^{-3}+j^{-4}+\cdots+j^{-r}+\cdots)$$

$$=\frac{j^{-2}}{1-j^{-1}}=\frac{1}{j(j-1)}=\frac{1}{j-1}-\frac{1}{j},$$

so that

$$12\sum_{j=2}^{k}f(j)<\left(1-\frac{1}{k}\right). \tag{2.4}$$

To eva'uate the sigma in (2.4) more accurately, we write

$$\sum_{j=2}^{k}f(j)=\sum_{j=2}^{+\infty}f(j)-\sum_{j=k+1}^{+\infty}f(j)=S_1-S_2;$$

and observe that

$$\frac{1}{j}-\frac{1}{j+1}<j^{-2}<\frac{1}{j-1}-\frac{1}{j},$$

so that

$$\sum_{j=k+1}^{\infty} j^{-2}=k^{-1}+O(k^{-2});$$

while

$$\sum_{j=k+1}^{\infty} j^{-r}=O(k^{-r+1}) \text{ for each } r>2.$$

Using the results on p. 811 of the Handbook, we get

$$S_1=0.08106\ 14667\ 952\cdots$$

while our observations show that

$$S_2=(12k)^{-1}+O(k^{-2}).$$

Hence

(2.5) $$\ln(k!)=\left(k+\frac{1}{2}\right)\ln k-k+0.91893\ 85332\ 047\cdots+(12k)^{-1}+O(k^{-2}).$$

REMARK. This is exactly what STIRLING's formula would give, since

$$0.91893\ 85332\ 047\cdots=\ln\sqrt{2\pi}.$$

From (2.1) and (2.5), we now have

(2.6) $$\ln\binom{k^2}{k}=\left(k-\frac{1}{2}\right)\ln k+k-1.41893\ 85332\ 047\cdots+(4k)^{-1}+O(k^{-2}).$$

In what follows, we write

$$B(k)=\left(k-\frac{1}{2}\right)\ln k+k-1.41893\ 85332\ 047\cdots+(4k)^{-1}.$$

Evidently then

(2.7) $$\Delta B(k)=B(k+1)-B(k)=\ln k+2+O(k^{-2}).$$

3. On $P(k)$. A theorem of ROSSER states that for each $j\geqq 1$.

$$p_j>j\ln j.$$

Hence

(3.1) $$\Delta P(k)=\ln p_{k+1}>\ln(k+1)+\ln\ln(k+1)$$
$$>\ln k+\ln\ln k.$$

Since for $k>1618$, $\ln\ln k>2$, we conclude that for $k>1618$, $P(k)$ increases more rapidly than $B(k)$. Since for $2<k\leqq 1618$, $B(k)-P(k)>0$, it follows that there exists a least $k=k_0>1618$ for which $P(k)-B(k)>0$, and this inequality continues to hold for each $k>k_0$.

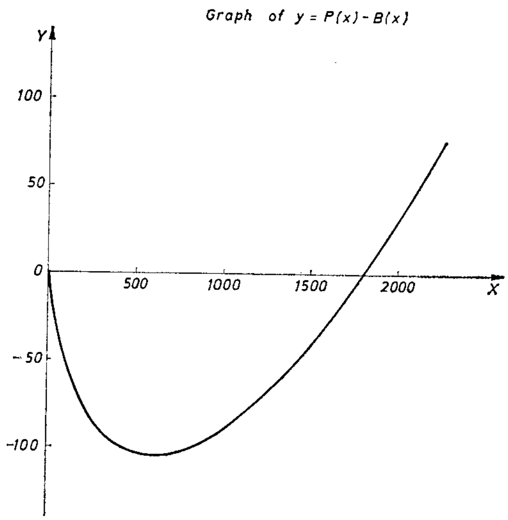
Our calculations presented in the next section show that $k_0 = 1794$.

Incidentally this shows that for $k \geqq 1794$, $\binom{n}{k}$ cannot have k or more distinct prime divisors for $n < k^2$. This greatly improves an earlier result of ERDÖS, GUPTA and KHARE [1].

We might here remark that roughly speaking $A(k) - B(k) = (6k^2)^{-1}$. For $k > 1618$, $A(k) - B(k)$ is therefore negligible.

4. The Table. The following short table and the graph to which it leads indicate the behaviour of $P(k) - B(k)$.

k	$P(k)$	$B(k)$	$P(k) - B(k)$
20	61.58629 51494	77.01034 08011	— 15.42404 56517
100	505.81623 31260	556.79799 49726	— 50.98176 18466
200	1180.52234 40312	1255.59662 60931	— 75.07428 20619
300	1917.04200 03144	2006.86474 59597	— 89.82274 56453
400	2693.54215 87642	2792.17177 30364	— 98.62961 42722
500	3499.34520 97226	3602.77830 66287	—103.43309 69061
600	4328.16837 91097	4433.54080 64355	—105.37242 73258
616	4462.68721 79927	4568.08997 33658	—105.40275 53731
617	4471.10988 07003	4576.51321 98909	—105.40333 91906
618	4479.53517 78770	4584.93808 84777	—105.40291 06007
700	5176.54385 42264	5281.06211 29725	—104.51825 87461
800	6041.39180 98372	6142.92845 02373	—101.53664 04001
900	6920.42102 99436	7017.33542 88548	— 96.91439 89112
1000	7812.28354 07328	7902.88271 28094	— 90.59917 20766
1100	8715.76090 04238	8798.45176 06752	— 82.69086 02514
1200	9629.32676 45305	9703.12843 43136	— 73.80166 97831
1300	10552.18688 44754	10616.15160 04873	— 63.96471 60119
1400	11484.08046 15577	11536.87764 81251	— 52.79718 65674
1500	12424.32382 59436	12464.75519 85754	— 40.43137 26318
1600	13371.76845 32844	13399.30659 14273	— 27.53813 81429
1700	14326.44932 74729	14340.11401 78359	— 13.66469 03630
1793	15220.29600 88819	15220.35571 23351	— 0.05970 34532
1794	15229.93546 57823	15229.84735 77569	0.08810 80254
1795	15239.57505 28910	15239.33956 07478	0.23549 21432
1800	15287.78079 32654	15286.80892 83754	0.97186 48900
1900	16255.08801 03246	16239.06380 22568	16.02420 80678
2000	17228.57909 34519	17196 58565 43212	31.99343 91307
2100	18207.64938 04085	18159.11084 39967	48.53853 64118
2200	19192.22019 27989	19126.40087 53456	65.81931 74533
2260	19785.50314 92770	19708.97102 05620	76.53212 87150



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