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578. ON SOME TRIANGLE INEQUALITIES*

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1. Introduction. In an attempt to obtain a certain system in the set of several hundreds of inequalities with respect to a triangle [1] VAN ALBADA remarked that the majority of them may be written as $P_n(a, b, c) \geqq 0$, where P_n is a homogeneous symmetric polynomial of order n for the variables a, b, c, representing the sides of a triangle [2]. He derived the complete set P_n for $n \leqq 3$ and gave some partial results for $n=4$. RIGBY [3] determined the complete sets for $n=2$ and $n=3$ by a simpler method. Of interest are the inequalities to be called *special*, with the property that equality holds for $a=b=c$.

In this paper we derive the complete set of special inequalities for $n=2$, $n=3$, $n=4$. In the sections 5 and 6 we discuss the relationship between these „algebraic" inequalities and some results involving either the sides or the elements s, R and r of the triangle, s being its semiperimeter, and R, r the radii of its circumscribed and inscribed circle.

2. A geometric mapping. The sides of a triangle satisfy the conditions that any one is less than the sum of the others. Therefore we introduce, as RIGBY did, instead of a, b, c the variables $u_1=-a+b+c$, $u_2=a-b+c$, $u_3=a+b-c$. This implies $2a=u_2+u_3$, $2b=u_3+u_1$, $2c=u_1+u_2$; it is obvious that the necessary and sufficient conditions are simply $u_i>0$ $(i=1, 2, 3)$.

A special inequality reads now $P_n(u_1, u_2, u_3) \geqq 0$, P_n being a homogeneous symmetric polynomial or order n, with the property $P_n(1, 1, 1)=0$.

To illustrate our method we consider u_i as the homogeneous triangle coordinates of a point with respect to an equilateral triangle $T=T_1T_2T_3$. The conditions $u_i>0$ express that the image point is inside T.

$P_n=0$ is the equation of a curve K_n of order n, which has the same symmetry as the equilateral triangle T. Furthermore $M(1, 1, 1)$, the centre of T, is a point of K_n and the symmetry implies that M is an isolated double point of K_n, with the isotropic lines through M as tangents.

Any symmetric polynomial of u_i is a function of the three elementary expressions:

$$(2.1) \qquad S_1=u_1+u_2+u_3, \quad S_2=u_2u_3+u_3u_1+u_1u_2, \quad S_3=u_1u_2u_3.$$

$S_1=0$ represents the line l at infinity, $S_2=0$ is the equation of the circumcircle of T, $S_3=0$ is that of the degenerate cubic consisting of the sides of T.

3. The cases $n=2$ and $n=3$. We have $P_2=\alpha S_1^2+\beta S_2$. If $P_2 \geq 0$ inside T then $P_2(1, 0, 0)=\alpha \geqq 0$; $P_2(1, 1, 1)=0$ implies $3\alpha+\beta=0$. The curve K_2 is a conic with an isolated double point at M and it is therefore degenerated into

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the two isotropic lines through M. Hence the *complete set of special inequalities of order 2 is given by*

(3.1) $$\alpha(S_1^2-3S_2)\geqq 0, \quad \alpha>0.$$

We have $P_3=\alpha S_1^3+\beta S_1S_2+\gamma S_3$. From $P_3(1,0,0)\geqq 0$ it follows $\alpha\geqq 0$; $P(1,1,1)=0$ implies $27\alpha+9\beta+\gamma=0$. Hence

(3.2) $$P_3=\alpha S_1^3+\beta S_1S_2-9(3\alpha+\beta)S_3.$$

We distinguish two cases $\alpha>0$ and $\alpha=0$.

If $\alpha>0$ we put $\beta=\alpha\beta_1$; the equation of K_3 is now

(3.3) $$S_1^3+\beta_1S_1S_2-9(\beta_1+3)S_3=0.$$

K_3 is a cubic curve with an isolated double point at M and therefore rational. It intersects l at the points at infinity of the sides. K_3 has no points inside T, different from M, if it does not intersect the sides of T at points between the vertices. The intersections with the sides are those on the circle $S_1^2+\beta_1S_2=0$, which for $\beta_1>-3$ is imaginary, for $\beta_1=-3$ the point circle at M, for $\beta_1=-4$ the inscribed circle and for $\beta=\infty$ the circumcircle. From this it follows that K_3 has no inside points (different from M) if $\beta\geqq -4$. We have found the following set of special cubic inequalities:

(3.4) $$S_1^3+\beta_1S_1S_2-9(\beta_1+3)S_3\geqq 0, \quad \beta_1\geqq -4.$$

For $\alpha=0$ we obtain

$$P_3=\beta(S_1S_2-9S_3).$$

The curve K_3 passes through the vertices T_i where it is tangent to the circumcircle; this implies that it has no points inside T, different from M. As $P_3(0,1,1)=2\beta$ we must have $\beta\geqq 0$; hence the inequality

(3.5) $$\beta(S_1S_2-9S_3)\geqq 0, \quad \beta\geqq 0.$$

From (3.4) and (3.5) it follows: *the complete set of special cubic inequalities is given by*

(3.6) $$\alpha S_1^3+\beta S_1S_2-9(\beta+3\alpha)S_3\geqq 0, \quad \alpha\geqq 0, \quad \beta\geqq -4\alpha.$$

Of the set (3.4) the best inequality is that for $\alpha=1$, $\beta_1=-4$; indeed we have $P_3(\beta_1)-P_3(-4)=(\beta_1+4)(S_1S_2-9S_3)\geqq 0$. In this case K_3 is tangent to the sides of T at their midpoints. The image curve K_3 is given in fig. 1 for $\alpha=1$, $\beta=-4$ and for $\alpha=0$ in fig. 2.

4. The case $n=4$. We have now

(4.1) $$P_4=\alpha S_1^4+\beta S_1^2S_2+\gamma S_1S_3+\delta S_2^2.$$

From $P_4(1,0,0)\geqq 0$ it follows $\alpha\geqq 0$. $P(1,1,1)=0$ implies $27\alpha+9\beta+\gamma+3\delta=0$. We distinguish once more two cases: $\alpha>0$, $\alpha=0$. If $\alpha>0$, $\beta=\alpha\beta_1$, $\delta=\alpha\gamma_1$, the equation of K_4 reads

(4.2) $$S_1^4+\beta_1S_1^2S_2-(27+9\beta_1+3\delta_1)S_1S_3+\delta_1S_2^2=0.$$

It is a quartic curve with an isolated double point at M; K_4 is tangent to l at the isotropic points. The condition $P_4\geqq 0$ for points inside T does not exclude that K_4 has more isolated double points in that region. We deal with this possibility later on. Apart from this case it is necessary that K_4 has no point

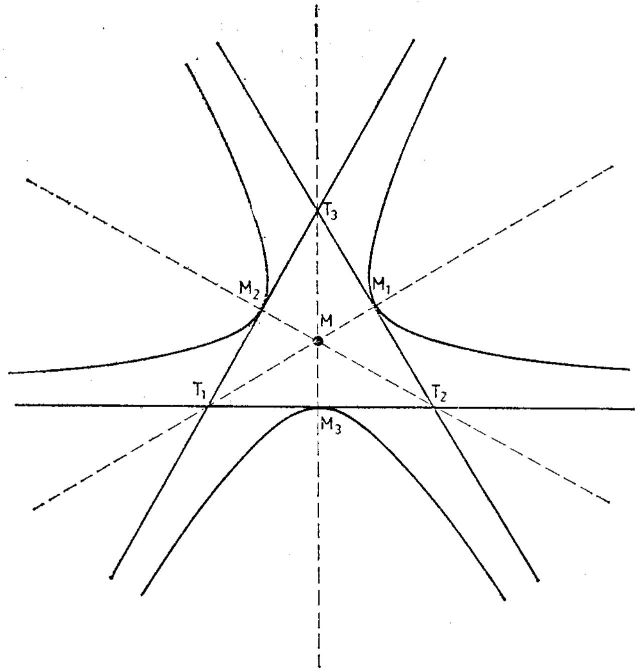

Fig. 1

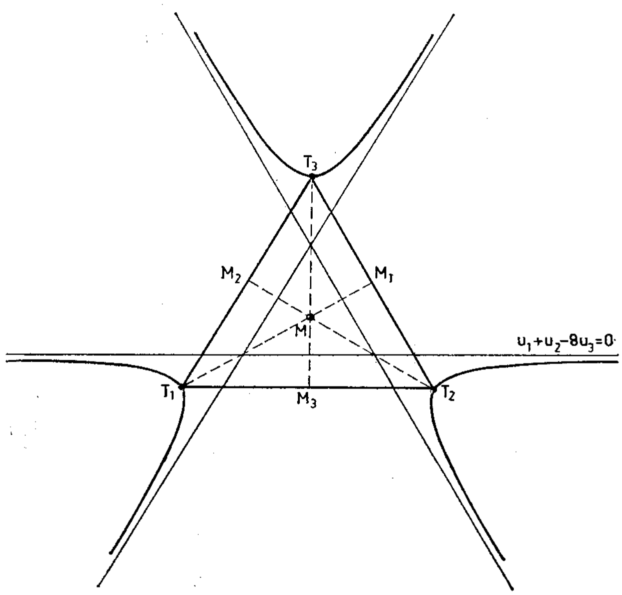

Fig. 2

between T_1 and M_1, M_1 being the midpoint of T_2T_3, and no point between T_2 and T_3. These conditions are also sufficient: they imply, in view of the curve's symmetry that it does not intersect the sides of the triangle T_2M_1M; any oval of K_4 inside this triangle, however, would once more in view of the symmetry, give rise to six ovals. This is impossible for a quartic curve; moreover a suitably chosen circle with centre M would have fourteen intersections with K_4. Hence our condition comes to this: K_4 has no point in common with either the intervals T_1M_1 or T_2T_3.

A parameter representation of the line T_1M_1 is $u_1=t$, $u_2=1$, $u_3=1$; points between T_1 and M_1 satisfy $t>0$. We have $S_1=t+2$, $S_2=2t+1$, $S_3=t$. The intersections with (4.2) are given by $t=1$ (twice) and by the roots of

$$t^2+2(5+\beta_1)t+16+4\beta_1+\delta_1=0, \tag{4.3}$$

a quadratic equation with discriminant $D=(\beta_1+3)^2-\delta_1$. The two roots are imaginary if $D<0$; for $D=0$ they coincide, being positive for $\beta_1<-5$, zero or negative for $\beta_1\geqq-5$; for $D>0$ the two roots are non-positive if $\beta_1\geqq 5$ and $4\beta_1+\delta_1+16\geqq 0$. In the (β_1, δ_1)-plane (fig. 3) we have the following situation: the condition is satisfied if the point (β_1, δ_1) is to the right of the border g consisting of the half-line g_1 along the line $\mathscr{L}$ with equation $4\beta_1+\delta_1+16=0$,

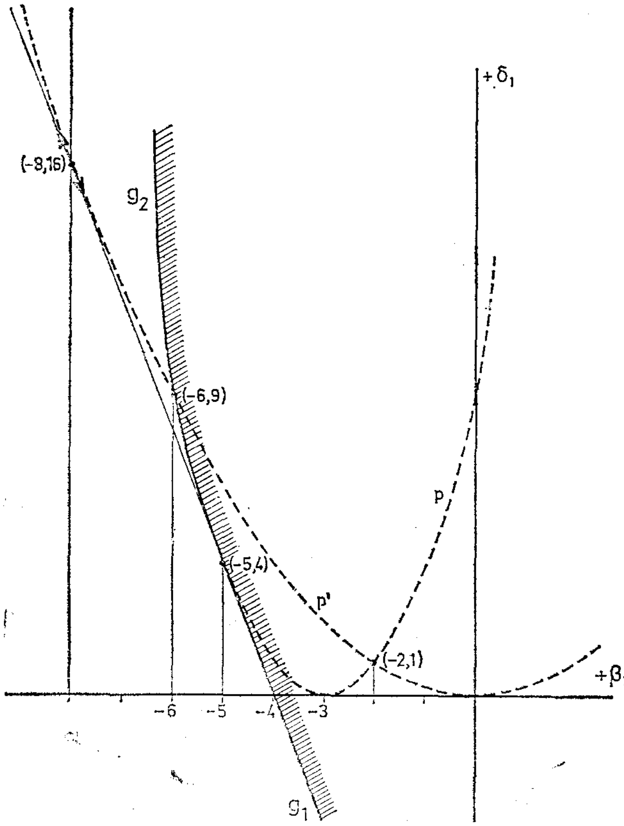

Fig. 3

from $Q(-5, 4)$ downwards, and of the arc g_2 of the parabola p with equation $\delta_1 = (\beta_1 + 3)^2$ from Q upwards; the line $\mathcal{L}$ is tangent to p at Q. Points on the border g will be considered later on.

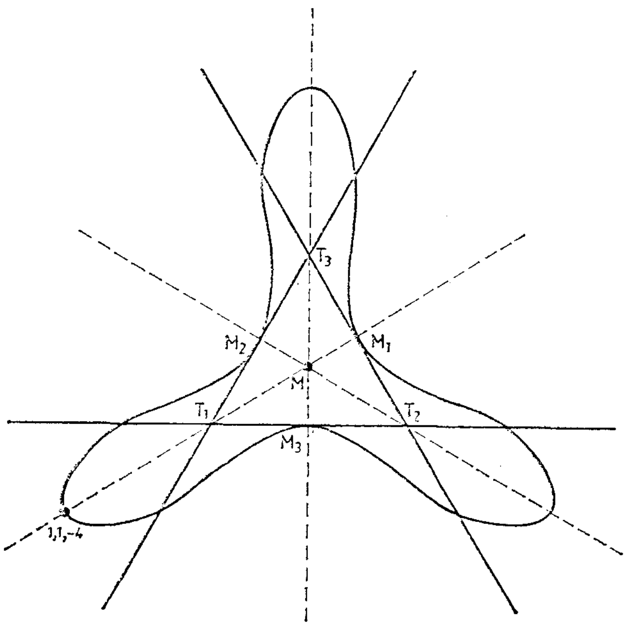

Fig. 4 ($\beta = -3$)

A further condition must be satisfied: K_4 has no intersections with the interval T_2T_3. Its intersection with $u_1 = 0$ are those of this line and

$$S_1^4 + \beta S_1^2 S_2 + \delta_1 S_2^2 = 0,$$

which represents two circles $S_1^2 - \lambda_1 S_2 = 0$ and $S_1^2 - \lambda_2 S_2 = 0$. A circle $S_1^2 - \lambda S_2 = 0$ is imaginary if $\lambda \leqq 3$, it is the inscribed circle of T if $\lambda = 4$ and the circumscribed circle if $\lambda = \infty$. Hence our condition comes to this: the roots of $\lambda^2 + \beta_1 \lambda + \delta_1 = 0$ must be either imaginary or both at most equal to 4. This implies that either $\beta_1^2 - 4\delta_1 < 0$ or $\beta_1^2 - 4\delta_1 \geqq 0$, $4\beta_1 + \delta_1 + 16 \geqq 0$, $\beta_1 + 8 \geqq 0$. It may be verified that all points (β_1, δ_1) on or to the right of g, satisfy these conditions. Thus we have found a set, with two parameters, of quartic special inequalities. It is given by (4.2) provided the point (β_1, δ_1) is in the region $\mathcal{G}$ of the (β_1, δ_1)-plane, where $\mathcal{G}$ consists of the points on or to the right of the border g. For $\delta_1 = 0$, K_4 is degenerated into l and a cubic curve; we find once more the cubic inequalities (3.4), the point $(-4, 0)$ being the intersection of $\mathcal{L}$ and the β_1-axis.

The most interesting inequalities will be those corresponding to points on the border g. On the lower part, we have $\delta_1 = -4\beta_1 - 16$; hence *we obtain the following one-parameter set of quartic inequalities*

(4.4) $\quad P_4 = S_1^4 + \beta_1 S_1^2 S_2 + 3(\beta_1 + 7) S_1 S_3 - 4(\beta_1 + 4) S_2^2 \geqq 0, \quad \beta_1 \geqq -5.$

One of the roots of (4.3) is now zero and K_4 passes through the midpoints M_i. The other root is $t = -2(\beta_1 + 5)$ and as $t = -2$ represents the point at infinity of $T_1 M_1$ we obtain for $\beta_1 > -4$ a quartic curve as given in fig. 4, for $-5 < \beta_1 < -4$ that of fig. 5 and for $\beta_1 = -4$ (the degenerate case) fig. 1. If $\beta_1 \to -5$ the three ovals of fig. 5 decrease and for $\beta_1 = -5$ they reduce to the points M_i. K_4 has four isolated points in this case and must therefore be degenerated. Its equation

(4.5) $$S_1^4 - 5\, S_1^2 S_2 + 6\, S_1 S_3 + 4\, S_2^2 = 0$$

may be written as $C_1 C_2 = 0$, with

$$C_1 = u_1^2 + \omega_1 u_1 u_2 + \omega_2 u_2^2 + u_2 u_3 + \omega_2 u_1 u_3 + \omega_1 u_3^2,$$

$$C_2 = u_1^2 + \omega_2 u_1 u_2 + \omega_1 u_2^2 + u_2 u_3 + \omega_1 u_1 u_3 + \omega_2 u_3^2,$$

where ω_i are the complex cubic roots of unity.

$C_1 = 0$ and $C_2 = 0$ represent two conics each passing through M, M_1, M_2, M_3 and tangent to l at an isotropic point. After some algebra it is seen that (4.5) is equivalent to

(4.6) $$\sum (-u_1 + u_2 + u_3)^2 \, (u_2 - u_3)^2 = 0.$$

The inequality for $\beta_1 = -5$ reads

(4.7) $$S_1^4 - 5\, S_1^2 S_2 + 6\, S_1 S_3 + 4\, S_2^2 \geqq 0,$$

with equality for the equilateral triangle (and for three degenerate triangles).

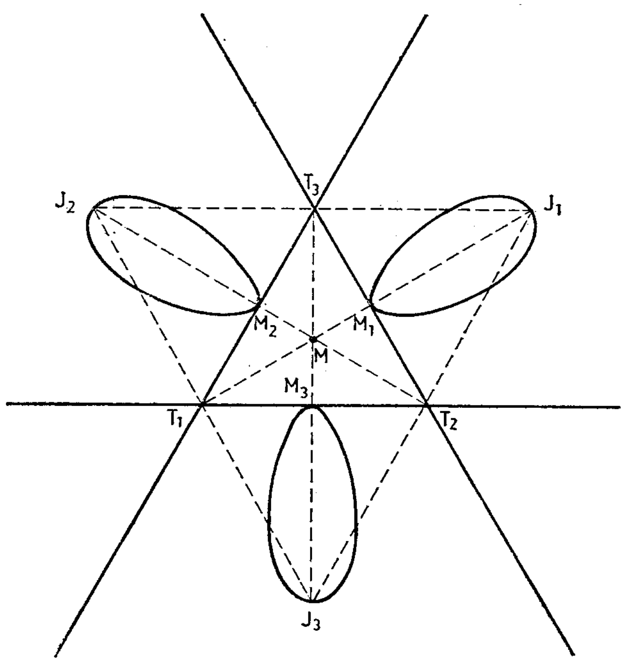

Fig. 5 $\left(\beta = -4\frac{1}{2}\right)$

We consider now the parabolic arc g_2. It is given by $\delta_1 = (\beta_1 + 3)^2$, $\beta_1 \leqq -5$. For the corresponding K_4 we obtain

$$(4.8) \qquad S_1^4 + \beta_1 S_1^2 S_2 - 3(\beta_1^2 + 9\beta_1 + 18) S_1 S_3 + (\beta_1 + 3)^2 S_2^2 = 0.$$

The equation (4.3) has two equal roots $t = -(\beta_1 + 5)$, corresponding to the point $\{-(\beta_1 + 5),\ 1,\ 1\}$ inside T. As could be expected and may be verified analytically this point and two analogous ones are isolated double points of (4.8) and its only real points. These points, at M_i for $\beta_1 = -5$, penetrate the triangle T if β decreases. For $\beta_1 = -6$ all three coincide with M; in this case the image point $(-6, 9)$ coincides with the left-hand intersection of the parabola p and the parabola p' with the equation $\delta_1 = \frac{1}{4}\beta_1^2$. K_4 is degenerated into two conics, for $\beta = -6$ both coinciding with the pair of isotropic lines through M; for $\beta_1 < -6$ the points are between M and T_i. We have found a *second one-parameter set of* special quartic inequalities:

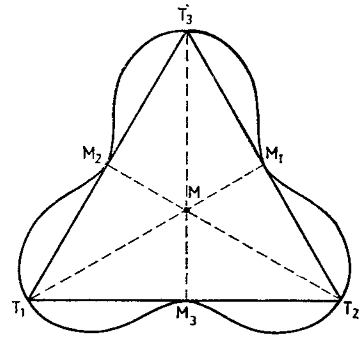

Fig. 6

$$(4.9) \quad S_1^4 + \beta_1 S_1^2 S_2 - 3(\beta_1^2 + 9\beta_1 + 18) S_1 S_3 + (\beta_1 + 3)^2 S_2^2 \geqq 0, \quad \beta_1 \leqq -5,$$

with equality for the equilateral triangle and for three isosceles triangles. The sets (4.4) and (4.9) have the inequality (4.7) in common. It is clear that (4.9) holds not only for points inside T but for all points of the u_i-plane. We consider now the case $\alpha = 0$ and obtain]

$$(4.10) \qquad P_4 = \beta S_1^2 S_2 - (9\beta + 3\delta) S_1 S_3 + \delta S_2^2.$$

K_4 passes now through the vertices T_i where it is tangent to the circumcircle. $P_4(0, 1, 1) \geqq 0$ implies $4\beta + \delta \geqq 0$. The intersections with $T_1 M_1$ follow from $(t-1)^2(2\beta t + 4\beta + \delta) = 0$; which gives $t = 1$ (twice) and $t = \infty$. For $\beta \neq 0$ the remaining root is $t = -\frac{4\beta + \delta}{2\beta}$; the condition for this point not to be an inside point reads $\beta > 0$. The intersections of K_4 and the sides of T follow from $\beta S_1^2 + \delta S_2 = 0$ and we know that they do not lie between the vertices if $-\delta/\beta < 4$. For $\beta = 0$ it is easily seen that K_4 has isolated double points at T_i and it is therefore degenerated into two conjugate, complex conics. *We have found the following set of special quartic inequalities*

$$(4.11) \qquad \beta S_1^2 S_2 - (9\beta + 3\delta) S_1 S_3 + \delta S_2^2 \geqq 0, \quad \beta \geqq 0, \quad 4\beta + \delta \geqq 0.$$

For $\delta = -3\beta$ the inequality is equivalent to (3.1).
We have in particular for $\beta = 0$,

$$(4.12) \qquad S_2^2 - 3 S_1 S_3 \geqq 0.$$

For $\beta > 0$ it follows from (4.11)

$$(4.13) \qquad S_1^2 S_2 - 3(\delta + 3) S_1 S_3 + \delta S_2^2 \geqq 0, \quad \delta \geqq -4$$

and still more special for $\delta = -4$:

(4.14) $$S_1^2 S_2 + 3 S_1 S_3 - 4 S_2^2 \geqq 0.$$

The curve K_4 corresponding to (4.13) is given in fig. 6.

We can show now that (4.7) is the best inequality of the set (4.4). Indeed we have, in view of (4.14), for (4.4):

$$P_4(\beta_1) - P_4(-5) = (\beta_1 + 5)\{S_1^2 S_2 + 3 S_1 S_3 - 4 S_2^2\} \geqq 0.$$

It is also the best inequality of the set (4.9): we have for the latter

$$P_4(\beta_1) - P_4(-5) = (\beta_1 + 5)\{S_1^2 S_2 - 3(\beta_1 + 4) S_1 S_3 + (\beta_1 + 1) S_2^2\}$$

and that is $\geqq 0$ in view of (4.13) for $\delta = \beta_1 + 1$.

In the same way it may be shown that (4.14) is the best inequality of the set (4.13).

5. Inequalities for the sides of the triangle. In the preceding sections the complete sets of special symmetric inequalities of order $n = 2$, 3 and 4 are given by means of the elementary symmetric functions S_1, S_2, S_3 of the positive numbers u_i. We can transform them in terms of the sides a, b, c of the triangle. Introducing

(5.1) $$s_1 = a + b + c, \quad s_2 = bc + ca + ab, \quad s_3 = abc,$$

it is easy to verify that the following relations hold:

(5.2) $$S_1 = s_1, \quad S_2 = -s_1^2 + 4 s_2, \quad S_3 = -s_1^3 + 4 s_1 s_2 - 8 s_3,$$

and conversely

(5.3) $$s_1 = S_1, \quad s_2 = \frac{1}{4}(S_1^2 + S_2), \quad s_3 = \frac{1}{8}(S_1 S_2 - S_3).$$

For $n = 2$ there is essentially only one inequality: (3.1) for $\alpha = 1$. By means of (5.2) we obtain

(5.4) $$s_1^2 - 3 s_2 \geqq 0,$$

a well-known result (it is left-hand side of G.I. **1.1**). For $n = 3$ the complete set is given by (3.6), which transforms into

(5.5) $$(7\alpha + 2\beta) s_1^3 - (27\alpha + 8\beta) s_1 s_2 + 18(3\alpha + \beta) s_3 \geqq 0, \quad \alpha \geqq 0, \quad \beta \geqq -4\alpha.$$

Any known special cubic inequality must be a member of this set. We mention some examples.

G.I. **1.2.** reads

$$35 s(a^2 + b^2 + c^2) - 36(s^3 + abc) \geqq 0,$$

or

(5.6) $$13 s_1^3 - 35 s_1 s_2 - 36 s_3 \geqq 0,$$

that is (5.5) for $\alpha = 17$, $\beta = -53$.

Another inequality is G.I. **1.3**:

$$abc - 8(s-a)(s-b)(s-c) \geqq 0,$$

or

(5.7) $$s_1^3 - 4 s_1 s_2 + 9 s_3 \geqq 0,$$

that is (5.5) for $\alpha = 0$, $\beta = \frac{1}{2}$.

G.I. **1.5.** gives us

$$8 (a^3 + b^3 + c^3) - 3 (b + c) (c + a) (a + b) \geqq 0,$$

or

(5.8) $$8 s_1^3 - 27 s_1 s_2 + 27 s_3 \geqq 0,$$

that is (5.5) for $\alpha = 5$, $\beta = -27/2$.

The rather complicated example G.I. **1.23.**

$$5 [bc (b + c) + ca (c + a) + ab (a + b)] - 3 abc - (a + b + c)^3 \geqq 0$$

may be written

(5.9) $$-s_1^3 + 5 s_1 s_2 - 18 s_3 \geqq 0,$$

which is (5.5) for $\alpha = 1$, $\beta = -4$; we recognize it as the best inequality of the set (3.4).

We consider now the case $n = 4$. The complete set derived above can be transformed into a two-parameter set of inequalities for a, b, c; any special symmetrie quartic inequality belongs to the set.

Let our first example be G.I. **1.14.**

$$b^2 c^2 + c^2 a^2 + a^2 b^2 - 2 abcs \geqq 0,$$

or

(5.10) $$s_2^2 - 3 s_1 s_3 \geqq 0$$

(a counter-part of (4.12) by the way), which transforms by means of (5.3) into

(5.11) $$S_1^4 - 4 S_1^2 S_2 + 6 S_1 S_3 + S_2^2 \geqq 0,$$

that is (4.2) for $\beta_1 = -4$, $\delta_1 = 1$; the image point $(-4, 1)$ is (fig. 3) in G and even on p, but not on g.

Another special symmetric quartic inequality (there are only a few in the G.I. collection) is G.I. **1.9.**

(5.12) $$abcs - a^3 (s - a) - b^3 (s - b) - c^3 (s - c) \geqq 0,$$

which may be shown to be equivalent to (4.12).

The best inequality of either the set (4.4) or the set (4.9) is (4.7) which may be transformed by means of (5.2) into the following strong inequality for the sides, which seems to be new:

(5.13) $$s_1^4 - 7 s_1^2 s_2 - 12 s_1 s_3 + 16 s_2^2 \geqq 0.$$

Another, which also could be unknown, follows from (4.14):

(5.14) $$-s_1^4 + 6 s_1^2 s_2 - 3 s_1 s_3 - 8 s_2^2 \geqq 0.$$

6. Inequalities for *s*, *R*, *r*. Many triangle inequalities are not expressed explicitly by means of the sides but contain other elements (goniometric functions of the angles, altitudes, medians, etc); they can always in principle be reduced to such where only the sides appear. We restrict ourselves to inequalities involving s, R and r. Use can be made of the following formulas [**4**]

$$S_1 = 2s, \quad S_2 = 4r(4R+r), \quad S_3 = 8r^2 s. \tag{6.1}$$

For $n=2$ formula (3.1) gives us

$$s^2 \geqq 3r(4R+r), \tag{6.2}$$

a well-known result (G.I. **5.5**; **5.6**) found as early as 1872.

For $n=3$ we obtain from (3.6), for $\alpha=1$,

$$s^2 \geqq (8\beta+27)r^2 - 4\beta Rr, \quad \beta \geqq -4 \tag{6.3}$$

also well-known [**5**], and for $\beta=-4$ Steinig's inequality

$$s^2 \geqq -5r^2 + 16Rr. \tag{6.4}$$

For $n=4$ some less known results are derived. For (4.7), the best inequality of both the sets (4.4) and (4.9), we obtain by means of (6.1):

$$F(s^2) = s^4 + (-20R+r)rs^2 + 4r^2(4R+r)^2 \geqq 0. \tag{6.5}$$

If we substitute for s^2 the value $s_0^2 = (16R-5r)r$ we have $F(s_0^2) = -12(R-2r) \leqq 0$ with equality only for the equilateral triangle. Hence $F(s^2)$ has two real zero's; from (6.5) it follows that s^2 is at least the largest of them:

$$s^2 \geqq \frac{r}{2}\{(20R-r) + [3(4R-5r)(12R+r)]^{1/2}\}, \tag{6.6}$$

which improves (6.4).

Inequality (4.14) gives rise to

$$s^2 \geqq \frac{r(4R+r)^2}{R+r}, \tag{6.7}$$

which is, however, weaker than (6.4).

REFERENCES

1. O. Bottema, R. Ž. Đorđević, R. R. Janić, D. S. Mitrinović, P. M. Vasić: *Geometric inequalities*. Groningen, 1969. This book will be denoted by G. I.

2. P. J. Van Albada: *Geometric inequalities and their geometry*. These Publications № **338** — № **352** (1971), 41—45.

3. J. F. Rigby: *A method of obtaining related triangle inequalities, with applications*. These Publications № **412** — № **460** (1973), 217—226.

4. O. Bottema: *Inequalities for R, r, and s*. These Publications № **338** — № **352** (1971), 27—36.

5. O. Bottema: l.c. formula (5.3).