YU ISSN 0522—8441

U N I V E R Z I T E T U B E O G R A Ḍ U

PUBLIKACIJE

ELEKTROTEHNIČKOG FAKULTETA

SERIJA:

MATEMATIKA I FIZIKA

№ 577—№ 598
(1977)

BEOGRAD

Univ. Beograd. Publ. Elektrotehn. Fak.
Ser. Mat. Fiz. № 577 — № 598 (1977), 3—10.

577. ADDENDA TO THE MONOGRAPH „ANALYTIC INEQUALITIES“ I*

Dragoslav S. Mitrinović and *Petar M. Vasić*

SCOPE OF THE ADDENDA

Inequalities are playing a very active role in mathematical analysis and number theory currently. The following monographs, published during the last fifteen years, made considerable contributions to this field**:

1. *Inequalities* by E. F. Beckenbach and E. Bellman (1961; second revised printing in 1965 and third revised printing in 1971);
2. *Differential-and Integral-Ungleichungen und ihre Anwendung bei Abschätzungs-und Eindeutigkeitsproblemen* by W. Walter (1964); *Differential and Integral Inequalites* by W. Walter (1970);
3. *Analytic Inequalities* by D. S. Mitrinović in cooreparation with P. M. Vasić (1970);
4. A series of expository papers entitled: *Inequalities* I (1967), II (1970), and III (1972) edited by O. Shisha;
5. *Finite Sections of Some Classical Inequalities* by H. S. Wilf (1970);
6. *Inequalities in Mechanics and Physics* by G. Duvaut and J. L. Lions (1975);
7. *Approximation of Functions of Several Variables and Imbedding Theorems* by S. M. Nikol'skiǐ (1975);
8. *Integral Inequalities and Theory of Nonlinear Vibrations* by A. N. Filatov and L. V. Šarova (1976) (in Russian).

As soon as the book *Analytic Inequalities* was completed, its author systematically continued to collecting material concerning inequalities which had been published up to 1970, but which had not been incorporated in his book.

However, a large number of papers have appeared since 1970, papers partly inspired by the aforementioned monographs, and probably even more so by the challenge of research in various branches of pure and applied mathematics, where inequalities are often the basis of important lemmas for proving various theorems or for approximating various functions. Some selections from this vast material are clearly needed.

---

* Presented May 16, 1977 by R. P. Boas and S. Kurepa.

** We might mention that books 1, 2, 3, 5, 6 and 7 were published by *Springer-Verlag*, book 4 by *Academic Press* and book 8 by *Nauka* (Moskow—SSSR).

Mitrinović contacted a number of competent mathematicians, asking them to answer certain questions in connection with inequalities, which would help him to prepare the second edition of *Analytic Inequalities* so that it would be as complete and useful as possible for the readers. The letter sent by Mitrinović reads:

In preparing the new edition of my book **Analytic Inequalities** *(Springer, 1970) I became faced with the problem of how to select the material which has (particularly during the last few years) accumulated to quite an extent. On one hand various generalizations of known inequalities are invented, and on the other conditions under which a known inequality holds are sharpened, which results in sharper inequalities. Again, in some papers the conditions are weakened in order that the inequality in question remain unaltered. In principle all that contributes to the advancement of the theory of inequalities.*

I believe, however, that there are certain exaggerations. If a generalized inequality does not yield attractive and simple inequalities besides the already known ones, it seems that such generalization is not worth much. Indeed, will look among the generalized inequalities for the one which is needed, and which appeared naturally in some problems? I even think that there must exist certain canonical (fundamental) inequalities, and that the inequalities which cannot be further simplified are the most essential ones. Nevertheless, one must not be apodictic and insist upon what should be investigated.

I would like to add something concrete. The famous inequalities of Markoff and Čebyšev are examples of simple inequalities with a deep value. I think that it is generally recognized that those inequalities themselves are essential and not their numerous generalizations. Naturally, the same is true for inequalities such as Cauchy's, Hölder's, Minkowski's, etc.

In fact I believe that the opinions of G. H. Hardy as given in his paper **Prolegomena to a chapter on inequalities** *(J. London Math. Soc.* **4**, 61—78, *and* **5**, 80 (1929)) *are still quite true.*

My views on this subject are not yet crystallized. I would therefore appreciate receiving your comments and suggestions regarding the problem I exposed. I would particularly be grateful if you could let me know of papers on inequalities which merit special attention, and also of papers which contain inequalities that can hardly be used owing to their over exaggerated artificial generality.

Your opinion on this particular problem (which is not confined only to inequalities) would be of great use to me.

Almost all of the mathematicians to whom the letter was sent replied. Their replies contained many useful ideas and suggestions for the new edition of the book. One characteristic of those replies is an almost total agreement with the text of the letter. Certain disagreements will also be useful when the final criteria for the selection of topics are being established.

Mitrinović would like to acknowledge those invaluable suggestions and he believes that it is not without interest to give here the names of certain mathematicians who took part in this questionnaire. The names are listed in alphabetical order and classified by the respective countries:

AUSTRALIA: K. Mahler, B. Mond.

AUSTRIA: E. Hlawka.

CANADA: J. Aczél, H. S. M. Coxeter, M. S. Klamkin, H. Schwerdtfeger.

EAST GERMANY: H. Sachs.

FRANCE: J. Dieudonné, M. Zamansky.

ENGLAND: A. Erdélyi, H. J. Godwin, J. C. P. Miller, H. P. Mulholland, A. Oppenheim.

HUNGARY: L. Losonczi, P. Turán.

INDIA: S. K. Chatterjea.

ITALIA: E. Baiada, S. Cinquini, L. Giuliano, B. Segre, G. Talenti.

POLAND: R. Gutowski, J. G. Krzyż. C. Olech, A. Turowicz.

ROMANIA: B. Crstici, P. Caraman.

SSSR: E. A. Anfertieva, I. E. Bazilevič, D. L. Berman, N. G. Čudakov, G. Drinfeld, E. Godunova, S. G. Mihlin, A. D. Myškis, S. L. Sobolev, V. S. Vidensky.

SWEDEN: H. Shapiro.

USA: A. R. Amir-Moéz, E. F. Beckenbach, J. Chandra, Ky Fan, A. M. Fink, G. Gasper, E. Grosswald, J. V. Herod, E. Hewitt, E. Hille, Y. L. Luke, O. L. Mangasarian, D. C. B. Marsh, G. Pólya, M. Skalsky, O. Taussky-Todd, H. Teufel, B. A. Troesch, A. Zygmund.

WEST GERMANY: E. Beck, E. Stark.

YUGOSLAVIA: G. Čupona, V. Devidé, S. Kurepa.

The new edition of *Analytic Inequalities* will not appear for a few years so it is therefore of interest to publish a series of articles devoted to inequalities which were omitted from the 1970 edition of *Analytic Inequalities* (such as CLARKSON's inequality, for example) as well as to the more important inequalities published since 1970 which refer to topics from that book. All these results will be commented on in detail and the necessary references supplied. Special attention will be paid to results published in journals which are not easily obtainable, or articles written in languages which are not widely read.

MITRINOVIĆ would welcome any comments on *Analytic Inequalities* or on this series of articles.

RART I: **The Clarkson Inequalities**

This expository paper refers to the CLARKSON inequalities and to some other related inequalities for norms, which were not mentioned in *Analytic Inequalities*.

Primarily we shall prove three auxiliary lemmas.

Lemma 1. *If* $1<p\leqq 2$, *then for every* $u\in[0,\ 1]$ *the inequality*

$$((1+u)^q+(1-u)^q)^{1/q}\leqq 2^{1/q}(1+u^p)^{1/p} \qquad \left(\frac{1}{p}+\frac{1}{q}=1\right) \tag{1}$$

is valid.

Proof. Let us start from the function f, defined by

$$f(v)=\left(\left(\frac{v+1}{2}\right)^q+\left(\frac{v-1}{2}\right)^q\right)^{p-1}-\frac{1}{2}v^p-\frac{1}{2} \qquad (v\geqq 1).$$

The derivative of this function is

$$f'(v)=\frac{p}{2}\left(\frac{\left(\left(\frac{v+1}{2}\right)^{q-1}+\left(\frac{v-1}{2}\right)^{q-1}\right)^{1/(q-2)}}{\left(\left(\frac{v+1}{2}\right)^q+\left(\frac{v-1}{2}\right)^q\right)^{1/(q-1)}}\right)^{q-2}-\frac{p}{2}v^{p-1}.$$

In the inequality between means of different orders

$$\left(\frac{PX^r+QY^r}{P+Q}\right)^{1/r}\geqq\left(\frac{PX^{r-1}+QY^{r-1}}{P+Q}\right)^{1/(r-1)} \qquad (P,\ Q,\ X,\ Y\geqq 0)$$

put

$$P=X=\frac{v+1}{2}, \qquad Q=Y=\frac{v-1}{2}, \qquad r=q-1,$$

we get $f'(v)\leqq 0$ for $v\geqq 1$. Since $f(1)=0$, we have $f(v)\leqq 0$ for $v\geqq 1$. Putting $v=1/u$, we get (1).

Lemma 2. *If* z *is a complex number such that* $|z|\leqq 1$, *the inequality*

$$|1+z|^q+|1-z|^q\leqq 2(1+|z|^p)^{q-1} \qquad \left(\frac{1}{p}+\frac{1}{q}=1\right) \tag{2}$$

holds for $1<p\leqq 2$.

Proof. If we put $z=re^{i\theta}$, inequality (2) becomes

$$|1+re^{i\theta}|^q+|1-re^{i\theta}|^q\leqq 2(1+r^p)^{q-1}.$$

Without difficulty it can be verified that for a function g, defined by

$$g(\theta)=|1+re^{i\theta}|^q+|1-re^{i\theta}|^q \qquad (0\leqq r\leqq 1),$$

$\max g(\theta)=g(0)=g(\pi)=|1+r|^q+|1-r|^q$ holds, so that (1) implies (2).

In Lemma 2 let us substitute $z=x/y$ if $|x|\leqq|y|$, or $z=y/x$ if $|x|\geqq|y|$; we get the following:

Lemma 3. *If* $x,\ y\in\mathbf{C}$ *and* $1<p\leqq 2$, *then*

$$|x+y|^q+|x-y|^q\leqq 2(|x|^p+|y|^p)^{q-1} \qquad \left(\frac{1}{p}+\frac{1}{q}=1\right). \tag{3}$$

Inequalities (1), (2) and (3) were obtained for the first time by J. A. CLARKSON [1]. The same paper also gives a proof by J. S. FRAME using TAYLOR series.

The proof of inequality (1) given here is a modification of the proof by S. L. SOBOLEV [2]. Another proof is by K. O. FRIEDRICHS [3]. On this inequality see the book [4] by E. HEWITT and K. STROMBERG.

Inequality (3) appeared in 1968 on the entrance examination of the École nationale des ponts et chaussées, Écoles nationales supérieures de l'aéronautique, du génie maritime, et des télécommunications. Proofs of this and of some others inequalities derived from it are given in Revue Math. Spéc. **79** (1968/69), 597—602.

Theorem 1. *If* x, y *are elements of the space* l_p *(or* L_p*), where* $p \geqq 2$ *and* $(1/p)+(1/q)=1$, *then the inequalities*

$$2(\|x\|^p+\|y\|^p)^{q-1} \leqq \|x+y\|^q+\|x-y\|^q, \tag{4}$$

$$2(\|x\|^p+\|y\|^p) \leqq \|x+y\|^p+\|x-y\|^p \leqq 2^{p-1}(\|x\|^p+\|y\|^p), \tag{5}$$

$$\|x+y\|^p+\|x-y\|^p \leqq 2(\|x\|^q+\|y\|^q)^{p-1} \tag{6}$$

hold. If $1<p\leqq 2$, *the converse inequalities are valid.*

Proof. Let $x=(x_1, x_2, \ldots)$ and $y=(y_1, y_2, \ldots)$ be elements of the space l_p. If $1<p\leqq 2$, by applying MINKOWSKI's inequality

$$\left(\sum|\alpha_i+\beta_i|^r\right)^{1/r} \geqq \left(\sum|\alpha_i|^r\right)^{1/r}+\left(\sum|\beta_i|^r\right)^{1/r} \qquad (0<r<1)$$

for $\alpha_i=|x_i+y_i|^p$, $\beta_i=|x_i-y_i|^p$, $r=p/q$ and inequality (3), we get

$$\begin{aligned}\left(\sum_{i=1}^{+\infty}|x_i+y_i|^p\right)^{q/p}+\left(\sum_{i=1}^{+\infty}|x_i-y_i|^p\right)^{q/p} &\leqq \left(\sum_{i=1}^{+\infty}(|x_i+y_i|^q+|x_i-y_i|^q)^{p/q}\right)^{q/p}\\ &\leqq \left(\sum_{i=1}^{+\infty}(2(|x_i|^p+|y_i|^p)^{q/p})^{p/q}\right)^{q/p}\\ &= \left(\sum_{i=1}^{+\infty}2^{p/q}(|x_i|^p+|y_i|^p)\right)^{q/p}\end{aligned}$$

i.e. inequality (4) for the case $x, y \in l_p$ and $1<p\leqq 2$.

The proof is similar in the case where $x, y \in L_p$.

If $p\geqq 2$ it is sufficient to substitute $\frac{x+y}{2}$ for x and $\frac{x-y}{2}$ for y.

Inequalities (4) and (6) are equivalent, which is to obtain if we make the transformations $x \to \frac{x+y}{2}$ and $y \to \frac{x-y}{2}$.

Now we proceed to the proof of inequality (5). The right-hand inequality follows from (6) and the inequality between means of different orders. The left-hand inequality is obtained from the right-hand inequality if we put

$$x \to \frac{x+y}{2}, \quad y \to \frac{x-y}{2}.$$

If $1<p\leqq 2$ *these inequalities hold in the opposite sense.*

The second inequality in (9) is in fact inequality (5). The first inequality in (9) was proved, according to O. HANNER [6] by A. BEURLING at a seminar in Uppsala in 1945, but this proof was not published. Therefore O. HANNER reproduced BEURLING's proof in [6].

REMARK. H. S. SHAPIRO applied the parallelogram inequality (the second inequality in (5)) to a problem related to approximation theory [10].

In monograph [11, p. 129 and pp. 140—142] by S. L. SOBOLEV the following result, due to V. R. PORTNOV, is mentioned.

Theorem 4. *Let measures* $\mu(x)$ *and* $\chi(y)$ *respectively be associated with manifolds* Ω_x *and* Ω_y. *Let us define, on spaces* L_p *and* L_q *the mixed norm as follows*:

$$\|\varphi(x, y)\|_{p, q}=\left(\int_{\Omega_x}\left(\int_{\Omega_y}|\varphi(x, y)|^q \,d\chi(y)\right)^{p/q} d\mu(x)\right)^{1/p}.$$

Then, when $r\geqq \max(p, q, p', q')$, *with* $\frac{1}{p}+\frac{1}{p'}=1$ *and* $\frac{1}{q}+\frac{1}{q'}=1$, *we have*

$$\left(\left\|\frac{\varphi+\psi}{2}\right\|_{p, q}^r+\left\|\frac{\varphi-\psi}{2}\right\|_{p, q}^r\right)^{1/r}\leqq\left(\frac{\|\varphi\|^{r'}+\|\psi\|^{r'}}{2}\right)^{1/r'},$$

where $\frac{1}{r}+\frac{1}{r'}=1$.

The proof of this inequality is analogous to SOBOLEV's proof in [2] and is founded on inequality (6). The uniform convexity of the unit sphere in an arbitrary space with mixed norm of the indicated type follows therefrom.

The following result is directly connected with inequality (9).

Theorem 5. *If* x *and* y *are in* $l_p(1<p\leqq 2)$, *then for* $0<b\leqq p-1$ *inequality*

$$\|x+y\|^2+b\|x-y\|^2\leqq 2\|x\|^2+2\|y\|^2$$

holds. The greatest possible value of b *is* $p-1$.

Theorem 6. *If* x *and* y *are in* l_p $(2\leqq p)$, *then for* $p-1\leqq b$ *the reversed inequality of Theorem* 5 *holds. In this case, the smallest possible value of* b *is* $p-1$.

Theorems 5 and 6 were proved by W. L. BYNUM and J. H. DREW ([12] and [13]) who motivated the following definitions [13]:

A BANACH space V is an upper weak parallelogram space (or an upper w. p. space) with constant b if b is a number such that the inequality

$$\|x+y\|^2+b\|x-y\|^2\geqq 2\|x\|^2+2\|y\|^2 \qquad (2\leqq p \text{ and } p\leqq b+1)$$

holds for each x and y in V. A lower w. p. space with constant b is a space V for which there is a number b such that the inequality of Theorem 5 is valid for each x and y in V.

REMARK. Space L_p is a set of all equivalence classes of complex measurable functions $t \mapsto x(t)$ $(0 \leqq t \leqq 1)$ such that the LEBESGUE integral $\int_0^1 |x(t)|^p \, dt$ is finite. The norm in space L_p is defined by

$$\|x\| = \left(\int_0^1 |x(t)|^p \, dt \right)^{1/p}.$$

In similar way is defined the space $L_p(S, \Sigma, \mu)$ for a measure space (S, Σ, μ).

Space l_p is the set of all complex numbers $x = (x_1, x_2, \dots)$ such that series $\sum_{i=}^{+\infty} |x_i|^p$ is convergent. The norm in this space is defined by

$$\|x\| = \left(\sum_{i=1}^{+\infty} |x_i|^p \right)^{1/p}.$$

REFERENCES

1. CLARKSON J. A.: *Uniformly convex spaces.* Trans. Amer. Math. Soc. **40**, 396—414 (1936).
2. SOBOLEV, S. L.: *On an inequality* (Russian). Uspehi Mat. Nauk (N. S.) **1**, No 3—4 (13—14), 197 (1946).
3. FRIEDRICHS, K. O.: *On Clarkson's inequalities.* Comm. Pure Appl. Math. **23**, 603—607 (1970).
4. HEWITT, E. and K. STROMBERG: *Real and Abstract Analysis.* Berlin-Heidelberg-New York 1965; see in particular pp. 223—226.
5. BOAS, R. P.: *Some uniformly convex spaces.* Bull. Amer. Math. Soc. **46**, 304—311 (1940).
6. HANNER, O.: *On the uniform convexity of L^p and l^p.* Arkiv for Matematik **3**, 239—244 (1955).
7. MORAWETZ, C. S.: *Two L^p inequalities.* Bull. Amer. Math. Soc. **75**, 1299—1302 (1969).
8. NEUMANN, J. VON and R. JORDAN: *On inner products in linear metric spaces.* Ann. of Math. **36**, 719—724 (1935).
9. CLARKSON, J. A.: *The von Neumann-Jordan constant for the Lebesgue spaces.* Ann. of Math. **38**, 114—115 (1937).
10. SHAPIRO, H. S.: *Regularity properties of the element of closest approximation.* Trans. Amer. Math. Soc. (to appear).
11. SOBOLEV, S. L.: *Introduction to the theory of cubature formulas* (Russian). Moscow 1974.
12. BYNUM, W. L. and J. H. DREW: *A weak parallelogram law for l_p.* Amer. Math. Monthly **79**, 1012—1015 (1972).
13. BYNUM, W. L.: *Weak parallelogram laws for Banach spaces.* Notices Amer. Math. Soc. **22**, A-175 (1975).