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575. SOME CONSIDERATIONS ON IYENGAR'S INEQUALITY AND SOME RELATED APPLICATIONS*

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K. S. K. IYENGAR ([1]), by means of geometrical consideration, has proved the following theorem:

Theorem A. *Let f be a differentiable function on* $[a, b]$ *and* $|f'(x)| \leqq M$. *Then*

$$\left| \int_a^b f(x)\, dx - \frac{1}{2}(b-a)\big(f(a)+f(b)\big) \right| \leqq \frac{M(b-a)^2}{4} - \frac{1}{4M}\big(f(b)-f(a)\big)^2.$$

In [2] the following generalization of the Theorem A is proved analiticaly:

Theorem B. *Let* $x \mapsto f(x)$ *be a differentiable function defined on* $[a, b]$ *and* $|f'(x)| \leqq M$ *for every* $x \in (a, b)$. *If* $x \mapsto p(x)$ *is an integrable function on* (a, b) *such that*

$$0 < c \leqq p(x) \leqq \lambda c \qquad (\lambda \geqq 1,\ x \in [a, b]),$$

the following inequality holds

$$\left| A(f; p) - \frac{1}{2}\big(f(a)+f(b)\big) \right| \leqq \frac{M(b-a)}{2} \cdot \frac{(\lambda+q)(1-q^2)+2(\lambda-1)q}{2\lambda(1+q)-(\lambda-1)(1+q^2)},$$

where A and q are defined by

$$A(f; p) = \frac{\int_a^b p(x) f(x)\, dx}{\int_a^b p(x)\, dx} \quad \textit{and} \quad q = \frac{|f(b)-f(a)|}{M(b-a)}.$$

In the above mentioned paper [2], Theorem B is generalized in the case where $x \mapsto f(x)$ is twice differentiable function defined on $[a, b]$, having the following properties $|f''(x)| \leqq M\big(\forall x \in (a, b)\big)$ and $f'(a) = f'(b)$.

In the special case where $p(x) \equiv 1$, the following result, is obtained

$$\left| \frac{1}{b-a} \int_a^b f(x)\, dx - \frac{1}{2}\big(f(a)+f(b)\big) \right| \leqq \frac{M(b-a)^2}{16} - \frac{1}{4M}\left(\frac{f(b)-f(a)}{b-a} - f'(a)\right)^2, \tag{1}$$

what is a generalization of the Theorem A.

In this paper we shall prove a more rigorous inequality in comparison with (1), with considerably weak conditions related to the function f. Also, a generalization of such a result will be done.

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Let's denote by $C_\alpha\,(0<\alpha\leqq 1)$ the space of continuous functions $x\mapsto f(x)$ which satisfy the LIPSCHITZ's condition of the order $\alpha\,(0<\alpha\leqq 1)$. That means, if there exists such a constant $M\,(=M(f)<+\infty)$ that

$$|f(x)-f(y)|\leqq M|x-y|^\alpha \qquad (\forall x,\, y\in[a,\, b]),$$

the function f belongs to space C_α.

Notice that the norm in the space C_α is often introduced by

$$\|f\|_\alpha=\|f\|_0+\sup_{\substack{x,\, y\in[a,\, b]\\ x\neq y}}\frac{|f(x)-f(y)|}{|x-y|^\alpha},$$

where $\|f\|_0=\max\limits_{x\in[a,\, b]}|f(x)|$.

Theorem 1. *Let the function $f:[a,b]\to\mathbf{R}$ satisfies the following conditions:*

1° $f^{(n-1)}\in C_\alpha$ *(with the constant M and $0<\alpha\leqq 1$);*

2° $f^{(k)}(a)=f^{(k)}(b)=0 \qquad (k=1,\, \ldots,\, n-1),$

where $n\in\mathbf{N}$[1].

Then, the inequality is valid

$$(2)\qquad \left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{1}{2}\big(f(a)+f(b)\big)\right|$$
$$\leqq\frac{M(b-a)^{\alpha+n-1}}{(\alpha+n)^{(n)}}\left\{\zeta^{\alpha+n-1}-\frac{q}{2}\big(1+(\alpha+n-1)(2\zeta-1)\big)\right\},$$

where ζ is the real root of the equation

$$\zeta^{\alpha+n-1}-(1-\zeta)^{\alpha+n-1}=q$$

and

$$q=\frac{(\alpha+n-1)^{(n-1)}}{M(b-a)^{\alpha+n-1}}\big(f(b)-f(a)\big),\ p^{(m)}=p\,(p-1)\ldots(p-m+1) \qquad (m\in\mathbf{N}).$$

Proof. According to the condition 1°, we have

$$-M(x-a)^\alpha\leqq f^{(n-1)}(x)-f^{(n-1)}(a)\leqq M(x-a)^\alpha$$

and

$$-M(b-x)^\alpha\leqq f^{(n-1)}(b)-f^{(n-1)}(x)\leqq M(b-x)^\alpha,$$

i. e.

$$-M(x-a)^\alpha\leqq f^{(n-1)}(x)\leqq M(x-a)^\alpha$$

and

$$-M(b-x)^\alpha\leqq f^{(n-1)}(x)\leqq M(b-x)^\alpha.$$

Using the condition (2) and the $(n-1)$ times successive integration of the last inequalities on (a,x) and (x,b), we get

$$f(a)-M\frac{(x-a)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}}\leqq f(x)\leqq f(a)+M\frac{(x-a)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}}$$

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[1] For $n=1$ it is considered that $f^{(0)}\equiv f$ and the condition 2° does not exist.

and

$$f(b)-M\frac{(b-x)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}}\leqq f(x)\leqq f(b)+M\frac{(b-x)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}},$$

i. e.

(3) $\max\{f(a)-F(x),\ f(b)-G(x)\}\leqq f(x)\leqq \min\{f(a)+F(x),\ f(b)+G(x)\},$

where

$$F(x)=M\frac{(x-a)^{\alpha+n-1}}{(\alpha+n-1)^{(n-1)}}\quad\text{and}\quad G(x)=M\frac{(b-x)^{\alpha+n-1}}{(n+\alpha-1)^{(n-1)}}.$$

With regard to

$$\min(\alpha,\beta)=\frac{1}{2}(\alpha+\beta-|\beta-\alpha|)\quad\text{and}\quad \max(\alpha,\beta)=\frac{1}{2}(\alpha+\beta+|\beta-\alpha|)\quad(\alpha,\beta\in\mathbf{R}),$$

inequalities (3) are reduced to

(4) $$-\frac{1}{2}F(b)(g(t)-|q+h(t)|)\leqq f\big(a+t(b-a)\big)-\frac{1}{2}\big(f(a)+f(b)\big)\leqq\frac{1}{2}F(b)(g(t)-|q-h(t)|),$$

where

$$g(t)=t^{\alpha+n-1}+(1-t)^{\alpha+n-1}\quad\text{and}\quad h(t)=t^{\alpha+n-1}-(1-t)^{\alpha+n-1}.$$

Basing on (4), we can deduce that the inequality

$$\frac{1}{2}(|q+h(t)|+|q-h(t)|)\leqq g(t),$$

holds, i. e.

(5) $$\max(|q|,\ |h(t)|)\leqq g(t).$$

Since $0\leqq|h(t)|\leqq g(t)$ $(0\leqq t\leqq 1)$ and

(6) $$k(n)=\min_{0\leqq t\leqq 1} g(t)=\begin{cases}1 & (n=1),\\ \dfrac{4}{2^{\alpha+n}} & (n>1),\end{cases}$$

it follows from (5)

(7) $$|q|\leqq k(n).$$

On the basis of the above, we conclude that the equation $q-h(t)=0$ has a unique real solution $t=\zeta\,(\in[0,1])$. Also, the equation $q+h(t)=0$ has a unique real solution $t=1-\zeta$.

Since

$$\int_0^1|q-h(t)|\,\mathrm{d}t=\int_0^1|q+h(t)|\,\mathrm{d}t=q(2\zeta-1)+\frac{2}{\alpha+n}\big(1-\zeta^{\alpha+n}-(1-\zeta)^{\alpha+n}\big),$$

integrating (4), we get (2).

Thus the Theorem 1 is proved.

Basing on the inequality (7), we can formulate the following result:

Theorem 2. *Let the function* $f: [a, b] \to \mathbf{R}$ *fulfils conditions as in Theorem* 1 *and let* $k(n)$ *be defined by* (6). *Then, for every* $n \in \mathbf{N}$,

$$\sup_{\substack{x, y \in [a, b] \\ x \neq y}} \frac{|f^{(n-1)}(x) - f^{(n-1)}(y)|}{|x-y|^{\alpha}} \geqq \frac{(\alpha+n-1)^{(n-1)}}{k(n)\,(b-a)^{\alpha+n-1}} |f(b) - f(a)|.$$

From the Theorem 1 immediately follows:

Theorem 3. *Let function* $f: [a, b] \to \mathbf{R}$ *has a continued derivation of the order* $n-1$ *and the bounded derivation of the order* n, *i.e.* $|f^{(n)}(x)| \leqq M \, (\forall x \in (a, b))$. *If* $f^{(k)}(a) = f^{(k)}(b) = 0$ $(k = 1, \ldots, n-1)$, *then the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x - \frac{1}{2}\big(f(a) + f(b)\big) \right| \leqq \frac{M(b-a)^n}{(n+1)!} \left\{ \zeta^n - \frac{q}{2}\big(1 + n(2\zeta - 1)\big) \right\},$$

holds, where ζ *is the real root of the equation*

$$\zeta^n - (1-\zeta)^n = q \quad \left(q = \frac{n!}{M(b-a)^n} \big(f(b) - f(a)\big) \right).$$

Using considerations, mentioned above, and solutions of Problems 168 and 169 in [3] (These two problems have been formulated and solved by D. D. Adamović) the following theorem can be proved:

Theorem 4. *Let the function* $f: [a, b] \to \mathbf{R}$ *fulfils conditions as in Theorem* 3.

Then, for every $n \in \mathbf{N}$, $c_n \in (a, b)$ *exists, such that*

$$|f^{(n)}(c_n)| \geqq \frac{2^{n-1}\, n!}{(b-a)^n} |f(b) - f(a)|. \tag{8}$$

This result is the solution of the problem stated by D. S. Mitrinović [4].

Remark. The inequality (8) can be represented in the integral form replacing $f(x)$ by $\int_a^x g(t)\,\mathrm{d}t$.

For $n=2$ and $\alpha=1$, the Theorem 1 is reduced to the following corollary:

Corollary 1. *Let the function* $f: [a, b] \to \mathbf{R}$ *fulfils conditions*

1° $|f'(x) - f'(y)| \leqq M|x-y| \qquad (\forall\, x, y \in [a, b])$;

2° $f'(a) = f'(b) = 0$.

Then

$$\left| \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x - \frac{1}{2}\big(f(a) + f(b)\big) \right| \leqq \frac{M(b-a)^2}{24} - \frac{1}{2} \cdot \frac{(f(b) - f(a))^2}{M(b-a)^2}. \tag{9}$$

Note that under the condition 1° (from the Corollary 1) and $f'(a)=f'(b)$, the inequality (9) is reduced to the inequality

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x-\frac{1}{2}\big(f(a)+f(b)\big)\right|\leqq\frac{M(b-a)^2}{24}-\frac{1}{2M}\left(\frac{f(b)-f(a)}{b-a}-f'(a)\right)^2,$$

which is evidently more rigorous than the inequality (1).

If, in the Theorem 3, we put $f(a)=f(b)=0\left(\Rightarrow\ q=0\ \Rightarrow\ \zeta=\frac{1}{2}\right)$, we get

$$\left|\int_a^b f(x)\,\mathrm{d}x\right|\leqq\frac{M(b-a)^{n+1}}{2^n(n+1)!}.$$

The last inequality represents a generalization of the inequality postulated in the Problem 121 [**5**] (see also [**6**]).

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