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574. ON A METHOD OF SOLVING OF CONDITIONAL CAUCHY EQUATIONS*

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1. Let $(G, +)$, $(H, +)$ and $(R, +, \cdot)$ be two abelian groups and an integral domain, respectively. In the sequel, unless explicitly stated otherwise, the letter f will denote a map of G into R or G into H, according as the operation of multiplication does occur in the equation considered or does not. Moreover, Z will always stand for the set $f^{-1}(\{0\})$ and $Z' := G \setminus Z$. By $(G : F)$ we shall denote the index of a subgroup $(F, +)$ of $(G, +)$.

In [5] (cf. also [2], [4] and [7]) the authors deal with the conditional functional equation of MIKUSIŃSKI

$$(1) \qquad f(x+y) \neq 0 \quad \text{implies} \quad f(x+y) = f(x) + f(y).$$

PL. KANNAPPAN and M. KUCZMA have solved in [12] the equation

$$(2) \qquad f(x+y) - af(x) - bf(y) \neq 0 \quad \text{implies} \quad f(x+y) = f(x) + f(y),$$

(under certain conditions regarding the characteristic of $(R, +, \cdot)$) which generalizes (1) and some so called alternative functional equations (see [6], [13], [14], [16] and [17][1]). J. DHOMBRES and the present author have investigated in [3] and [4], among others, the following conditional equations

$$(3) \qquad f(x) + f(y) \neq 0 \quad \text{implies} \quad f(x+y) = f(x) + f(y)$$

and

$$(4) \qquad f(x+y) \neq 0 \quad \text{and} \quad f(x) + f(y) \neq 0 \quad \text{implies} \quad f(x+y) = f(x) + f(y).$$

J. ACZÉL's problem [1] may be easily reduced (cf. [11]) to the problem of finding the solutions of

$$(5) \qquad f(x+y) f(x) f(y) \neq 0 \quad \text{implies} \quad f(x+y) = f(x) + f(y)$$

(partial answers to that question are given in [11] and [10]). Observe that in the case where f maps G into R equation (4) may be written in the form

$$(4') \qquad f(x+y)\,[f(x) + f(y)] \neq 0 \quad \text{implies} \quad f(x+y) = f(x) + f(y).$$

Consequently, in that case, all the above mentioned equations are of the form

$$(\mathrm{E}) \qquad \varphi\big(f(x), f(y), f(x+y)\big) \neq 0 \quad \text{implies} \quad f(x+y) = f(x) + f(y),$$

where $\varphi : R^3 \to R$ is a given function. Observe that in (1), (2) and (3) φ is an affine function while it is not the case for (4′) and (5). The difference seems to be essential

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[1] A certain more general equation than (2) has been investigated in [9].

if the behaviour of solutions is taken into account. Namely, the results obtained in [5] and [12] state that, roughly speaking, equations (1) and (2) do not admit non--additive solutions provided the cardinality of $f(G)$ is greater than three. Similarly, as was shown in [8] (cf. also an earlier result in [3] obtained under an additional assumption on the range of f), non-additive solutions of (3) take values in a set of at most two elements. The situation changes if we deal with equations (4′) and (5); in [4] (cf. also [3]) and [11] the authors exhibit non-additive solutions of (4′) and (5), respectively, with $f(G)$ infinite. Bearing these remarks in mind one might conjecture that for an arbitrary affine function φ equation (E) admits additive solutions only provided the cardinality of $f(G)$ is greater than three. We shall show (Theorem 2 below) that this is really the case for a centroaffine function φ: $\varphi(u, v, w) = au + bv + cw$ and that the conjecture fails for $\varphi(u, v, w) = au + bv + cw + d$, $d \neq 0$ (see **4** below). On the other hand, our simple Theorem 1 allows to obtain a unified approach to the question of solving certain equations of the form (E).

2. In this section we do not assume the commutativity of the given groups $(G, +)$ and $(H, +)$; the integral domain $(R, +, \cdot)$ will not occur. Let us start with the following

Theorem 1. *Assume* $\varphi : H^3 \to H$ *to be a given function and suppose that a function* $f : G \to H$ *is a solution of* (E). *Put* $F(x, y) := \varphi\big(f(x), f(y), f(x+y)\big)$, $x, y \in G$. *Then* f *is additive if and only if*

$$\text{(C)} \qquad \begin{cases} \textit{for every pair } (x, y) \in G^2 \textit{ such that } f(x+y) \neq f(x) + f(y) \textit{ there exists} \\ \textit{a } z \in G \textit{ such that } F(x+y, z) \neq 0 \textit{ and } F(x, y+z) \neq 0 \textit{ and } F(y, z) \neq 0. \end{cases}$$

Proof. The "only if" part is trivial. Assume (C) and suppose, for the indirect proof, that f is not additive, i.e. there exists a pair $(x, y) \in G^2$ such that $f(x+y) \neq f(x) + f(y)$. Consequently, on account of (C) and (E), there exists a $z \in G$ such that

$$f(x+y+z) = f(x+y) + f(z), \quad f(x+y+z) = f(x) + f(y+z)$$

and

$$f(y+z) = f(y) + f(z).$$

Hence

$$f(x+y) + f(z) = f(x) + f(y+z) = f(x) + f(y) + f(z)$$

and thus

$$f(x+y) = f(x) + f(y),$$

contrary to our supposition.

Examples of applicability:

1° MIKUSIŃSKI's equation (1). Suppose that f is a non-additive solution of (1). Here $F(x, y) = f(x+y)$. On account of Theorem 1 we have non (C), i.e. there exists a pair $(x, y) \in G^2$ such that $f(x+y) \neq f(x) + f(y)$ and

$$(6) \qquad f(x+y+z) = 0 \quad \text{or} \quad f(y+z) = 0, \quad \text{for all} \quad z \in G.$$

Relation (6) says that $(-y-x+Z) \cup (-y+Z) = G$ or, equivalently, $Z \cup (x+Z) = G$. Hence $Z' \subset x+Z$. Observe that $Z' \neq \varnothing$ (otherwise f would be the zero-function and hence additive). Taking an $s \in Z'$ we get $s = x+z$, $f(z) = 0$, $0 \neq f(s) = f(x+z) = f(x) + f(z) = f(x)$, i.e. $f|_{Z'} = f(x) \neq 0$. Moreover, we have $Z' + Z' \subset Z$. In fact, take $s, t \in Z'$; if we had $s + t \in Z'$, then

$$0 \neq f(x) = f(s+t) = f(s) + f(t) = 2f(x)$$

whence $f(x)=0$, a contradiction. On the other hand, (1) implies $Z+Z \subset Z$. Recalling Lemma 2 from **[8]** we infer that $(Z, +)$ is a subgroup of index 2 in $(G, +)$.

Corollary 1. (cf. the main result in **[5]**). *Let f be a soltuion of* (1). *Then f is additive or*

$$f(x)=\begin{cases} 0 & \text{for } x \in Z \\ d \neq 0 & \text{for } x \in Z' \end{cases} \tag{7}$$

where Z is such that $(Z, +)$ *is a subgroup of* $(G, +)$ *with* $(G : Z)=2$ *and* $d \in H \setminus \{0\}$ *is an arbitrary constant. Conversely, every such function as well as every additive function is a solution of* (1).

2° DHOMBRES's equation (3). Suppose that f is a non-additive solution of (3). Here $F(x, y)=f(x)+f(y)$. On account of Theorem 1, we have non (C), i. e. there exists a pair $(x, y) \in G^2$ such that $f(x+y) \neq f(x)+f(y)$ (whence $f(x)+f(y)=0$ and $f(x+y) \neq 0$) and

$$f(x+y)+f(z)=0 \quad \text{or} \quad f(x)+f(y+z)=0 \quad \text{or} \quad f(y)+f(z)=0,$$

for all $z \in G$.

If $f(z) \notin \{-f(y), \ -f(x+y)\}=\{f(x), \ -f(x+y)\}$, then $0=f(x)+f(y+z)$ $=f(x)+f(y)+f(z)=f(z)$. Consequently

$$f(z) \in \{0, \ f(x), \ -f(x+y)\}=: T \text{ for all } z \in G. \tag{8}$$

Note that f is odd (it suffices to put one of the variables equal to the inverse of the other in (3)). Consequently (3) may be written in the form

$$f(s) \neq f(t) \quad \text{implies} \quad f(s-t)=f(s)-f(t). \tag{9}$$

At first, we shall exclude the possibility: card $f(G)=3$. In fact, otherwise (8) gives $f(G)=T$ with $0 \neq f(x) \neq -f(x+y) \neq 0$. Since $f(x+y) \in T$ we infer that

(i) $f(x+y)=f(x)$ or (ii) $f(x+y)=-f(x+y)$.

Assume (i). Then $0 \neq f(y)=-f(x) \neq f(x)$ and (9) gives the equality $f(y-x)$ $=f(y)-f(x)=2f(y)=f(x)$ (by (3)). Hence

$$0 \neq 2f(y)=f(2y)=f(y-x)+f(x+y)=2f(x)=f(y),$$

a contradiction. Assuming (ii), in view of $f(x)+f(x+y) \neq 0$, we infer that

$$f(x)+f(x+y)=f(2x+y) \in T \setminus \{0\},$$

whence by (8) and the fact that $f(x+y) \neq 0$, $f(x)=0$ and, again, we have come to a contradiction. Thus card $f(G) \leqq 2$. Since, by our assumption, f is non-additive, this leads to $f(x)=d \neq 0$, $2d=0$, $x \in G$, or $f(G)=\{0, d\}$, $d \neq 0=2d$. In the latter case we have also $Z+Z' \subset Z'$ whence, by Lemma 1 from **[8]**, we infer that $(Z, +)$ is a group. Thus we have come to the following

Corollary 2. *Let f be a solution of* (3). *Then f is additive or* $f(x)=d \neq 0=2d$, $x \in G$, *or f is of the form* (7) *where Z is such that* $(Z, +)$ *is a subgroup of* $(G, +)$ *and* $H \ni d \neq 0$ $=2d$. *Conversely, each of the above type function yields a solution of* (3).

(Equation (3) has been solved in **[3]** under the additional assumption that $(H, +)$ does not possess non-zero elements of order 2; cf. also **[8]**).

3° The conditional CAUCHY equation

(10) $$f(x)\neq f(y) \quad \text{implies} \quad f(x+y)=f(x)+f(y).$$

Assume f to be a solution of (10). It is readily seen that

$$f(0)=0 \quad \text{or} \quad f=\text{const}\neq 0.$$

Suppose that f is non-constant and non-additive. Then $Z\neq\varnothing$ and $Z'\neq\varnothing$. Moreover, (10) gives immediately $Z+Z'\subset Z'$ whence, by Lemma 1 from **[8]**, $(Z, +)$ is a group. Here $F(x, y)=f(x)-f(y)$ and, on account of Theorem 1, we have non (C), i. e. there exists a pair $(x, y)\in G^2$ such that $f(x+y)\neq f(x)+f(y)$ (whence $f(x)=f(y)$) and

$$f(z)=f(x+y) \quad \text{or} \quad f(x)=f(y+z) \quad \text{or} \quad f(z)=f(y), \quad \text{for all} \quad z\in G.$$

If $f(z)\notin\{f(y), f(x+y)\}$, then $f(y)+f(z)=f(y+z)=f(x)=f(y)$ whence $f(z)=0$. Consequently

$$f(z)\in\{0, f(x), f(x+y)\}=:T \quad \text{for all} \quad z\in G.$$

Let us consider two cases:

(i) card $T=2$. Then f is of the form (7).

(ii) card $T=3$. Then $0\neq f(x)\neq f(x+y)\neq 0$. In particular, $f(x)+f(x+y)=f(2x+y)\in T$ which implies $f(x+y)=-f(x)$. Thus $T=\{0, d, -d\}$, $d\neq 0$, $d\neq -d$. Take an $s\in f^{-1}(\{d\})=:Z_d$ and $t\in f^{-1}(\{-d\})=:Z_{-d}$. We shall show that

(11) $$Z\cup(Z-s)\cup(Z-t)=G.$$

In fact, otherwise, one can find a $z\in G$ such that $f(z)\neq 0$, $f(z+s)\neq 0$ and $f(z+t)\neq 0$. Suppose $z\in Z_d$; then $f(z)\neq f(t)$ and $0\neq f(z+t)=f(z)+f(t)=d-d=0$, a contradiction. Analogously, if we had $z\in Z_{-d}$ then $f(z)\neq f(s)$ and $0\neq f(z+s)=f(z)+f(s)=-d+d=0$, a contradiction. Relation (11) may equivalently be written in the form

$$Z\cup(s+Z)\cup(t+Z)=G,$$

whence, in view of the inclusions: $s+Z\subset Z_d$ and $t+Z\subset Z_{-d}$, we get $Z_d=s+Z$ and $Z_{-d}=t+Z$. This means that $(G:Z)=3$ and

(12) $$f(x)=\begin{cases} 0 & \text{for} \quad x\in Z \\ d & \text{for} \quad x\in Z_1 \\ -d & \text{for} \quad x\in Z_2, \end{cases}$$

where Z_1, Z_2 denote the cosets of Z. Thus we have come to the following

Corollary 3. *Let f be a solution of* (10). *Then f is additive or $f=$* const *or f is of the form* (7) *where Z is such that $(Z, +)$ is a subgroup of $(G, +)$, or f is of the form* (12) *where Z is such that $(Z, +)$ is a subgroup of $(G, +)$ with $(G:Z)=3$, Z_1, Z_2 are the cosets of Z and $d\in H\setminus\{0\}$ is an arbitrary constant.*

(Under several additional assumptions (10) has been investigated in **[12]**; cf. also **[15]**).

3. Now, we are going to solve the functional equation of the type (E) with an arbitrary centroaffine function $\varphi: R^3\to R$ in the class of functions $f: G\to R$ where $(G, +)$ is a commutative group and $(R, +, \cdot)$ is an arbitrary integral domain (with no restrictions on its characteristic). The explicit form of our conditional CAUCHY equation reads as follows:

(13) $af(x)+bf(y)+cf(x+y)\neq 0$ implies $f(x+y)=f(x)+f(y)$.

Its solutions are described by the following

Theorem 2. *Let $f: G\to R$ be a solution of* (13). *Then the following cases are the only possible ones;*

(i) *$a=b=c=0$ and f is arbitrary;*

(ii) *a, b, c are arbitrary and f is additive;*

(iii) *$a+b+c=0$ and f is constant;*

(iv) *$b=-a$, $c=0$ and f is of the form* (7) *where $(Z, +)$ is a subgroup of $(G, +)$ with $(G:Z)>2$ and $d\in R\setminus\{0\}$;*

(v) char $(R, +, \cdot)=2$, *$b\neq -a$, $c=a-b$ and f is as in* (iv) *with $2d=0$;*

(vi) *$b=-a$, c is arbitrary and f is of the form* (7) *where $(Z, +)$ is a subgroup of $(G, +)$ with $(G:Z)=2$ and $d\in R\setminus\{0\}$;*

(vii) *$a=b=0$, $c\neq 0$ and f is as in* (vi);

(viii) *$c=a+b$ and f is of the form* (12) *where $(Z, +)$ is a subgroup of $(G, +)$ with $(G:Z)=3$ and $d\in R$ with $2d\neq 0$.*

Conversely, each of the above type function yields a solution of (13).

Proof. Suppose f to be a non-constant and non-additive solution of (13). Putting $y=0$ in (13) we infer that

$$(a+c)f(x)\neq -bf(0) \quad \text{implies} \quad f(0)=0, \qquad x\in G.$$

If we had $f(0)\neq 0$, then $(a+c)f(x)=bf(0)$, $x\in G$, and in case $b\neq 0$ we would have $a+c\neq 0$ and $f=$ const, contrary to our assumption; in case $b=0$ we come to $c=-a$ and equation (13) assumes the form

$$af(x)-af(x+y)\neq 0 \quad \text{implies} \quad f(x+y)=f(x)+f(y).$$

Putting here $x=0$ we get $a\,[f(0)-f(y)]=0$, $y\in G$. Then either $a=0$ and we have case (i) or $f(y)=f(0)$, $y\in G$, a contradiction. Thus, in the sequel, we may assume

(14) $$f(0)=0.$$

Next, we shall show that f satisfies (10). For, interchanging x and y in (13) and subtracting we get easily the following relation

$$f(x+y)\neq f(x)+f(y) \quad \text{implies} \quad (a-b)[f(x)-f(y)]=0,$$

which is simply (10) provided $a\neq b$. Assume $a=b$. Then $a=0$ and $c=0$ leads to (i) whereas $a=0$ and $c\neq 0$ reduces (13) to MIKUSINSKI's equation (1) and we have case (vii) (cf. Corollary 1). The conjunction $a\neq 0$ and $c=-a$ is evidently excluded; consequently, in what follows, we may assume

(15) $$b=a\neq 0 \quad \text{and} \quad a+c\neq 0.$$

Observe that f must be odd; in fact, putting $y=-x$ (13) we get in view of (14) and (15) that $f(-x)=-f(x)$, $x\in G$. Take a pair $(x, y)\in G^2$ such that

$$f(y)\neq f(x+y)-f(x)=f(x+y)+f(-x).$$

On account of (13) we get

$$a[f(x+y)-f(x)]+cf(y)=0$$

as well as

$$a[f(x)+f(y)]+cf(x+y)=0.$$

Adding the last two equalities side by side we obtain

$$(a+c)[f(x+y)+f(y)]=0,$$

whence, by virtue of (15), we have $f(x+y)+f(y)=0$. Since the roles of x and y are symmetric, we have also $f(x+y)+f(x)=0$ and, consequently, $f(x)=f(y)$. This proves that f satisfies (10) in case (15), too. According to Corollary 3, f is of the form (7) where $(Z, +)$ is a subgroup of $(G, +)$ and $d \in R \setminus \{0\}$ or f is of the form (12) where $(Z, +)$ is a subgroup of index 3 in $(G, +)$, Z_1, Z_2 are the cosets of Z and $d \in R \setminus \{0\}$. Evidently, (10) need not be equivalent to (13) and we must check whether these functions yield solutions of (13). Regarding the first possibility we shall distinguish two cases: $(G:Z)>2$ and $(G:Z)=2$. In the first one, taking $x, y \in Z'$ such that $x+y \in Z$ we get

$$(a+b)\, d \neq 0 \quad \text{implies} \quad 0=2\,d,$$

whereas for $x, y \in Z'$ such that $x+y \in Z'$ we obtain

$$(a+b+c)\, d \neq 0 \quad \text{implies} \quad d=2\,d. \tag{16}$$

Thus, $b=-a$ implies $c=0$ and hence case (iv) whereas $b \neq -a$ gives $2\,d=0$ and, consequently, $\operatorname{char}(R, +, \cdot)=2$, as well as $c=-a-b$, i. e. case (v). In the case where $(G:Z)=2$ relation (16) disappears and, since $2\,d \neq 0$ (otherwise, f would be additive), we get $b=-a$ with no restrictions on c, that is case (vi).

Regarding the possibility that f is of the form (12), take $x, y \in Z_1$; then $x+y \in Z_2$ whence

$$(a+b-c)\, d \neq 0 \quad \text{implies} \quad -d=2\,d.$$

Since $3\,d=0$ is excluded (otherwise, f would be additive) we infer that $c=a+b$ coming to case (viii).

Finally, it is readily seen that every additive function $f: G \to R$ satisfies (13) independently of the values of a, b and c (case (ii)) as well as that a constant function $G \ni x \mapsto f(x)=d \in R \setminus \{0\}$ is a solution of (13) if and only if $a+b+c=0$ (case (iii)).

The last part of the theorem is obvious. Thus our proof has been completed.

REMARK 1. Bearing in mind the solutions of (3) (cf. Corollary 2) one may ask why some of them are not included in cases occurring in the statement of Theorem 2. Indeed, the case where $a=b=1$, $c=0$ and f is of the form (7) with $(Z, +)$ being a subgroup of index greater than 2 in $(G, +)$ and $d \neq 0=2d$, does not occur explicitly in Theorem 2. Observe, however, that the condition $d \neq 0=2d$ implies $\operatorname{char}(R, +, \cdot)=2$ whence $0=c=-a-b=-2\,a$, $b=-a=1$, i. e. we have case (iv). Similarly, the constant solution $f(x)=d$, $x \in G$, of (3) with $d \neq 0=2d$ does occur if and only if $\operatorname{char}(R, +, \cdot)=2$; then we have $a+b+c=2\,a=0$ in (13) and the solution is involved in (iii).

REMARK 2. Our proof of Theorem 2 does not require the knowledge of solutions of DHOMBRES's equation (3).

REMARK 3. Theorem 2 generalizes the main result from **[12]** not only in that c is not assumed to be a unit of $(R, +, \cdot)$ (in particular, DHOMBRES's equation is not involved in (2)) but also in that the characteristic of the integral domain $(R, +, \cdot)$ may be quite arbitrary. The authors of **[12]** assume the characteristic zero (in genuine, their proof method requires the characteristic to be different from 2 and 3; this fact is explicitly underlined in the final remark in **[12]**).

4. Now, we are going to exhibit an example showing that if the function φ in Theorem 1 is affine but not centroaffine, then equation (E) admits nonadditive solutions whose range is of cardinality greater than three (in fact — infinite). For, take $(G, +)=(\mathbf{R}, +)$ — the additive group of all real numbers and $(R, +, \cdot)=$ $=(\mathbf{R}, +, \cdot)$ — the integral domain of all real numbers with the usual addition and multiplication. Define $\varphi : \mathbf{R}^3 \to \mathbf{R}$ by the formula

$$\varphi(u, v, w) := w - u - v - 1, \qquad (u, v, w) \in \mathbf{R}^3.$$

The corresponding functional equation

$$f(x+y) \neq f(x)+f(y)+1 \quad \text{implies} \quad f(x+y)=f(x)+f(y) \tag{17}$$

is evidently satisfied by a non-additive function $\mathbf{R} \ni x \mapsto f(x) := [x]$ (entier of x). The general solution of (E) with an affine but not centroaffine function φ is not known to me even in the case of equation (17).

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