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553. SOME NUMBERS RELATED TO THE STIRLING NUMBERS OF THE FIRST AND SECOND KIND*

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1. The STIRLING numbers of the first and second kind are defined by

$$x(x+1)\cdots(x+n-1)=\sum_{k=0}^{n} S_1(n,k)\,x^k \tag{1.1}$$

and

$$x^n=\sum_{k=0}^{n} S(n,k)\,x(x-1)\cdots(x-k+1), \tag{1.2}$$

respectively. They satisfy the recurrences

$$S_1(n+1,\ k)=S_1(n,\ k-1)+nS_1(n,\ k), \tag{1.3}$$

$$S(n+1,\ k)=S(n,\ k-1)+kS(n,\ k). \tag{1.4}$$

It is not difficult to show that $S_1(n,n-k)$ and $S(n,n-k)$ are polynomials in n of degree $2k$. Moreover

$$S_1(n,\ n-k)=\sum_{j=0}^{k-1} S_1'(k,\ j)\binom{n}{2k-j} \qquad (k>0) \tag{1.5}$$

and

$$S(n,\ n-k)=\sum S'(k,\ j)\binom{n}{2k-j} \qquad (k>0). \tag{1.6}$$

The coefficients $S_1'(k,\ j)$, $S'(k,\ j)$ were introduced by JORDAN [**10**, Ch. 4] and WARD [**12**]. We are using the notation of [**6**]. See also [**13**].

The Eulerian numbers $A_{n,k}$ are usually defined by ([**3**], [**11**, Ch. 8])

$$\frac{1-\lambda}{1-\lambda\, e^{x(1-\lambda)}}=1+\sum_{n=1}^{\infty}\frac{x^n}{n!}\sum_{k=1}^{n} A_{n,k}\,\lambda^k. \tag{1.7}$$

They satisfy the recurrence

$$A_{n+1,k}=(n-k+2)\,A_{n,k-1}+kA_{n,k} \tag{1.8}$$

as well as

$$x^n=\sum_{k=1}^{n} A_{n,k}\binom{x+k-1}{n}. \tag{1.9}$$

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The representation (1.9) suggests that it may be of interest to consider representations of this kind for $S_1(n, n-k)$ and $S(n, n-k)$. Some general results on such representations are discussed in some detail in [8] but will not be needed. We shall show that

$$S_1(n, n-k) = \sum_{j=1}^{k} B_1(k, j)\binom{n+j-1}{2k} \tag{1.10}$$

and

$$S(n, n-k) = \sum_{j=1}^{k} B(k, j)\binom{n+j-1}{2k}. \tag{1.11}$$

Moreover we find that

$$B_1(k, j) = B(k, k-j+1) \qquad (1 \leq j \leq k). \tag{1.12}$$

We shall also show that

$$B_1(k, j) = a_{k,j}, \tag{1.13}$$

where the $a_{k,j}$ are numbers that first occurred in an entirely different connection [4].

Combinatorial interpretations of the $S_1(n, k)$, $S(n, k)$, $S_1'(n, k)$, $S'(n, k)$ are known [6], [11, Ch. 4]. However no combinatorial interpretation of $B_1(n, k)$ or $B(n, k)$ is known.

2. Since $S_1(n, n-k)$ is a polynomial in n of degree $2k$, we may put

$$S_1(n, n-k) = \sum_{j=0}^{2k} B_1(k, j)\binom{n+j-1}{2k}. \tag{2.1}$$

However it follows from (1.5) that $B_1(k, 0) = 0$, $B_1(k, j) = 0$ $(k < j \leq 2k)$. Thus (2.1) reduces to

$$S_1(n, n-k) = \sum_{j=1}^{k} B_1(k, j)\binom{n+j-1}{2k}. \tag{2.2}$$

By (1.3),

$$S_1(n+1, n-k+1) = S_1(n, n-k) + nS_1(n, n-k+1). \tag{2.3}$$

Thus

$$\sum_{j=1}^{k} B_1(k, j)\binom{n+j}{2k} - \sum_{j=1}^{k} B_1(k, j)\binom{n+j-1}{2k} = n\sum_{j=1}^{k-1} B_1(k-1, j)\binom{n+j-1}{2k-2}$$

$$= \sum_{j=1}^{k-1} B_1(k-1, j)\binom{n+j-1}{2k-2}\big((n-2k+j+1) + (2k-j-1)\big)$$

$$= \sum_{j=1}^{k-1} B_1(k-1, j)\left\{(2k-1)\binom{n+j-1}{2k-1}\right.$$

$$\left. + (2k-j-1)\left(\binom{n+j}{2k-1} - \binom{n+j-1}{2k-1}\right)\right\},$$

so that

$$\sum_{j=1}^{k} B_1(k, j)\binom{n+j-1}{2k-1} = (2k-1)\sum_{j=1}^{k-1} B_1(k-1, j)\binom{n+j-1}{2k-1}$$

$$+\sum_{j=2}^{k}(2k-j)B_1(k-1, j-1)\binom{n+j-1}{2k+1} - \sum_{j=1}^{k-1}(2k-j-1)B_1(k-1, j)\binom{n+j-1}{2k-1}.$$

It follows that

(2.4) $$B_1(k, j) = (2k-j)B_1(k-1, j-1) + jB_1(k-1, j) \qquad (1 \leqq j \leqq k).$$

Since $S_1(n, n-1) = \binom{n}{2}$, it is evident that

(2.5) $$B_1(1, 1) = 1, \quad B_1(1, j) = 0 \qquad (j > 1).$$

The numbers $B_1(k, j)$ are completely determined by (2.4) and (2.5).

The following table is easily computed.

$B_1(k, j)$:

k \ j	1	2	3	4	5
1	1				
2	1	2			
3	1	8	6		
4	1	22	58	24	
5	1	52	328	444	120

Turning next to $S(n, k)$, we have

(2.6) $$S(n, n-k) = \sum_{j=1}^{k} B(k, j)\binom{n+j-1}{2k}.$$

By (1.4),

(2.7) $$S(n+1, n-k+1) = S(n, n-k) + (n-k-1)\, S(n, n-k+1),$$

so that

$$\sum_{j=1}^{k} B(k, j)\binom{n+j}{2k} - \sum_{j=1}^{k} B(k, j)\binom{n+j-1}{2k}$$

$$= \sum_{j=1}^{k-1} B(k-1, j)\binom{n+j-1}{2k-2}\big((n-2k+j+1) + +(k-j)\big)$$

$$= (2k-1)\sum_{j=1}^{k-1} B(k-1, j)\binom{n+j-1}{2k-1}$$

$$+\sum_{j=1}^{k-1}(k-j)B(k-1, j)\left(\binom{n+j}{2k-1} - \binom{n+j-1}{2k-1}\right).$$

It follows that

(2.8) $$B(k, j) = (k-j+1)B(k-1, j-1) + (k+j-1)B(k-1, j) \qquad (1 \leqq j \leqq k).$$

Since $S(n, n-1) = \binom{n}{2}$ we hawe

(2.9) $$B(1,1) = 1, \quad B(1, j) = 0 \qquad (j > 1).$$

The $B(k, j)$ are completely determined by (2.8) and (2.9).

3. In (2.8) replace j by $k-j+1$ and we get

(3.1) $$B(k, k-j+1) = jB(k-1, k-j) + (2k-j)B(k-1, k-j+1) \qquad (1 \leqq j \leqq k).$$

If we put $\bar{B}(k, j) = B(k, k-j+1)$, (3.1) becomes

(3.2) $$\bar{B}(k, j) = j\bar{B}(k-1, j) + (2k-j)B(k-1, j-1) \qquad (1 \leqq j \leqq k).$$

Also, by (2.9),

(3.3) $$\bar{B}(1, 1) = 1, \quad \bar{B}(1, j) = 0 \qquad (j > 1).$$

Therefore, comparison with (2.4) and (2.5) yields

(3.4) $$B_1(k, j) = B(k, k-j+1) \qquad (1 \leqq j \leqq k).$$

An explicit formula formula for $B(k, j)$ can be found in the following way. Multiply both sides of (2.6) by x^n and sum over n. Since

$$\sum_{n=k}^{+\infty} x^{n-k} \sum_{j=1}^{k} B(k, j)\binom{n+j-1}{2k} = \sum_{j=1}^{k} B(k, j)\, x^{k-j+1} \sum_{n=k}^{+\infty} \binom{n+j-1}{2k} x^{n-2k+j-1}$$

$$= \sum_{j=1}^{k} B(k, j)\, x^{k-j+1} \sum_{n=0}^{+\infty} \binom{n+2k}{2k} x^n = (1-x)^{-2k-1} \sum_{j=1}^{k} B(k, j)\, x^{k-j+1}$$

$$= (1-x)^{-2k-1} \sum_{j=1}^{k} B(k, k-j+1)\, x^j$$

we get

(3.5) $$\sum_{j=1}^{k} B(k, k-j+1)\, x^j = (1-x)^{2k+1} \sum_{t=0}^{+\infty} S(t+k, t)\, x^t \qquad (k > 0).$$

Comparison of coefficients gives

(3.6) $$B(k, k-j+1) = \sum_{t=0}^{j} (-1)^{j-t} \binom{2k+1}{j-t} S(t+k, t)$$

and therefore

(3.7) $$B(k, k-j+1) = \sum_{t=0}^{j} (-1)^{j-t} \binom{2k+1}{j-t} \frac{1}{t!} \sum_{s=0}^{t} (-1)^{t-s} \binom{t}{s} s^{t+k}.$$

Analogous to (3.6), we hawe also

(3.8) $$B_1(k, k-j+1) = \sum_{t=0}^{j} (-1)^{j-t} \binom{2k+1}{j-t} S_1(t+k, t).$$

In view of (3.4) we may state

(3.9) $$B(k, j) = \sum_{t=0}^{j} (-1)^{j-t} \binom{2k+1}{j+t} S_1(t+k, t).$$

By (1.11) and (3.9),

$$S(n, n-k) = \sum_{j=1}^{k} \binom{n+j-1}{2k} \sum_{t=0}^{j} (-1)^{j-t} \binom{2k+1}{j-t} S_1(t+k, t)$$

$$= \sum_{t=0}^{k} S_1(t+k, t) \sum_{j=t}^{k} (-1)^{j-t} \binom{2k+1}{j-t} \binom{n+j-1}{2k}.$$

The inner sum on the extreme right can be summed by SAALCHÜTZ's theorem [1, p. 9] and we find, after some manipulation, that

(3.10) $$\sum_{j=1}^{k} (-1)^{j-t} \binom{2k+1}{j-t} \binom{n+j-1}{2k} = \binom{k+n}{k-t} \binom{k-n}{k+t}.$$

Therefore we have

(3.11) $$S(n, n-k) = \sum_{t=0}^{k} \binom{k+n}{k-t} \binom{k-n}{k+t} S_1(t+k, t).$$

To express $S_1(n, k)$ in terms of $S(n, k)$, we take

$$S_1(n, n-k) = \sum_{j=1}^{k} B_1(k, j) \binom{n+j-1}{2k} = \sum_{j=1}^{k} B(k, k-j+1) \binom{n+j-1}{2k}$$

$$= \sum_{j=1}^{k} \binom{n+j-1}{2k} \sum_{t=0}^{j} (-1)^{j-t} \binom{2k+1}{j-t} S(t+k, t)$$

$$= \sum_{t=0}^{k} S(t+k, t) \sum_{j=t}^{k} (-1)^{j-t} \binom{2k+1}{j-t} \binom{n+j-1}{2k}.$$

Hence, by (3.10),

(3.12) $$S_1(n, n-k) = \sum_{t=0}^{k} \binom{k+n}{k-t} \binom{k-n}{k+t} S(t+k, t).$$

Note that the coefficients on the right of (3.11) and (3.12) are the same. The formulas (3.11) and (3.12) are not new; for references see [7] and [9].

4. While we have obtained an explicit formula for $B(k, j)$ in (3.7), we have been unable to find a simple generating function. However the following results are of some interest in this connection.

Put

4.1) $$e^{nz} = \sum_{r=0}^{n} \frac{(nz)^r}{r!} + \frac{(nz)^n}{n!} S_n(z),$$

where n is a positive integer and z is an arbitrary complex number. BUCKHOLTZ [2] proved that, for $k \geq 1$,

$$S_n(z) = \sum_{r=0}^{k-1} n^{-r} U_r(z) + O(n^{-k}),$$

uniformly in a certain region of the z-plane. The coefficients $U_r(z)$ are determined by

$$U_r(z) = (-1)^r \left(\frac{z}{1-z}\frac{d}{dz}\right)^r \frac{z}{1-z}. \tag{4.2}$$

It follows from (4.2) that

$$U_r(z) = (-1)^r (1-z)^{-2r-1} Q_r(z), \tag{4.3}$$

where, for $r \geqq 1$, $Q_r(z)$ is a polynomial of degree r with positive integral coefficients.

The writer [4] showed that, if we put

$$Q_k(z) = \sum_{j=1}^{k} a_{kj} z^j \qquad (k \geqq 1), \tag{4.4}$$

then the a_{kj} satisfy the recurrence

$$a_{kj} = j a_{k-1,j} + (2k-j)\, a_{k-1,j-1} \qquad (1 \leqq j \leqq k). \tag{4.5}$$

Moreover

$$a_{11} = 1, \quad a_{ij} = 0 \qquad (j > 1). \tag{4.6}$$

Therefore, comparing (4.5) with (3.2), we have at once

$$a_{kj} = B_1(k, j) = B(k, k-j+1). \tag{4.7}$$

Put $f_j(x) = \sum_{k=j}^{\infty} a_{kj} x^k$ $(j = 1, 2, \ldots)$. Then, by (4.5),

$$f_j(x) = \sum_{k=j}^{\infty} \left(j a_{k-1,j} + (2k-j)\, a_{k-1,j-1}\right) x^k$$

which gives

$$(1-jx) f_j(x) = x\left(2x\frac{d}{dx} - j + 2\right) f_{j-1}(x) \qquad (j > 1). \tag{4.8}$$

For example, since $f_1(x) = x/(1-x)$, $f_2(x) = \dfrac{2x^2}{(1-x)^2(1-2x)}$, so that

$$a_{k2} = 2^{k+1} - 2k - 2 \qquad (k = 0, 1, 2, \ldots). \tag{4.9}$$

Generally, it follows from (4.8) that

$$f_j(x) = \frac{x^j P_j(x)}{(1-x)^j (1-2x)^{j-1} \cdots (1-jx)} \qquad (j = 1, 2, \ldots), \tag{4.10}$$

where $P_j(x)$ is a polynomial in x of degree $< \frac{1}{2} j(j-1)$, $j > 1$, that satisfies the recurrence

$$P_j(x) = (1-x)\cdots(1-(j-1)x)\left\{2x\frac{d}{dx} + 2\sum_{s=1}^{j-1}\frac{s(j-s)x}{1-sx} + j\right\} P_{j-1}(x) \quad (j > 1). \tag{4.11}$$

For example, since $P_1(x)=1$, we hawe

$$P_2(x)=(1-x)\left(2x\frac{d}{dx}+\frac{2x}{1-x}+2\right)\cdot 1=2,$$

$$P_3(x)=(1-x)(1-2x)\left(2x\frac{d}{dx}+\frac{4x}{1-x}+\frac{4x}{1-2x}+3\right)\cdot 2=6-2x-6x^2.$$

It follows from (4.10) or otherwise that

$$a_{kj}=\sum_{s=0}^{j}\Phi_{j,s}(k)(j-s)^k, \tag{4.12}$$

where $\Phi_{j,s}(k)$ is a polynomial in k of degree s. Indeed, by (3.7),

$$B(k,k-j+1)=\sum_{s=0}^{j}s^k\sum_{t=s}^{j}(-1)^{j-t}\binom{2k+1}{j-t}\binom{t}{s}\frac{s^t}{t!}$$

$$=\sum_{s=0}^{j}(j-s)^k\sum_{t=j-s}^{j}(-1)^{j-t}\binom{2k+1}{j-t}\binom{t}{s}\frac{s^t}{t!},$$

so that

$$\Phi_{j,s}(k)=\sum_{t=0}^{s}(-1)^t\binom{2k+1}{t}\binom{j-t}{s}\frac{s^t}{(j-t)!}. \tag{4.13}$$

Using (3.7) we get

$$a_{k2}=B(k,k-1)=2^{k+1}-2k-2,$$

$$a_{k3}=B(k,k-2)=\frac{1}{2}3^{k+2}-(2k+2)2^{k+1}+\frac{1}{2}+\binom{2k+2}{3}.$$

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