UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 544 — № 576 (1976), 39–42.

551. EQUIVALENT L_p-NORMS OF SUBADDITIVE FUNCTIONS*

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In this paper we prove the following theorem which is an extension of a result of R. P. GOSSELIN [3]. GOSSELIN's result is essentially the special case $F(x)=|x|^{n+p\alpha}$, and the following proof is a modification of that in [3].

Theorem. *Let $\varphi:\mathbf{R}^n\to\mathbf{R}$ be positive, measurable, and subadditive, and let $0<p<+\infty$. Let F be positive and measurable on $\mathbf{R}^n$ and satisfy the following condition: if $\omega_0,\omega_1,\omega_2$ are unit vectors in $\mathbf{R}^n$ such that $\omega_0=r_1\omega_1+r_2\omega_2$ for some $r_1, r_2\in\mathbf{R}$ then*

$$(1)\qquad F(r\,\omega_0)\geqq K(|r_i|)\,F(rr_i\,\omega_i)\ \text{for}\ r\geqq 0,\ i=1,2,$$

where $K:\mathbf{R}^+\to\mathbf{R}^+$ is measurable, $\inf\{K(r):\ \alpha\leqq r\leqq\beta\}>0$ for $0<\alpha<\beta<+\infty$, and $\int\limits_0^1 r^{-n}\,\mathrm{d}r/K(r)<+\infty$. Let $\omega_1,\ldots,\omega_n$ be linearly independent unit vectors, and set

$$(2)\qquad M(\omega)=\int\limits_0^{+\infty}\frac{\varphi^p(r\,\omega)\,r^{n-1}}{F(r\,\omega)}\,\mathrm{d}r.$$

There exist constants A, B depending on K, p, n such that

$$(3)\qquad \int\limits_{\mathbf{R}^n}\frac{\varphi^p(x)}{F(x)}\,\mathrm{d}x\leq A\sum_{i=1}^{n}\max\,[M(\omega_i),\ M(-\omega_i)]\leq B\int\limits_{\mathbf{R}^n}\frac{\varphi^p(x)}{F(x)}\,\mathrm{d}x.$$

By taking $r_1=1$, $r_2=0$ we observe that (1) implies $K(1)\leqq 1$. For GOSSELIN's result, (1) holds with $K(s)\equiv s^{-(n+p\alpha)}$. To prove the theorem we write $x=r\,\omega$, where

$$\omega=(\cos\varphi_1,\ \sin\varphi_1\cos\varphi_2,\ \ldots,\ \sin\varphi_1\cdots\sin\varphi_{n-2}\cos\varphi_{n-1},\ \sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{n-1}),$$

and let

$$\Omega=\{\omega\in\mathbf{R}^n:|\omega|=1\}=\{\omega:0\leqq\varphi_j\leqq\pi,\ 1\leqq j\leqq n-2,\ 0\leqq\varphi_{n-1}\leqq 2\pi\}.$$

With $\mathrm{d}\omega=\sin\varphi_{n-2}\sin^2\varphi_{n-3}\cdots\sin^{n-2}\varphi_1\,\mathrm{d}\varphi_1\cdots\mathrm{d}\varphi_{n-1}$, we have [**2**, p. 268]

$$\int\limits_{\mathbf{R}^n} f(x)\,\mathrm{d}x=\int\limits_{\Omega}\Big(\int\limits_0^{+\infty} f(r\,\omega)\,r^{n-1}\,\mathrm{d}r\Big)\,\mathrm{d}\omega.$$

Using this notation, we now appeal to GOSSELIN's Lemma [3]:

* Presented June 29, 1976 by D. S. MITRINOVIĆ.

There exist constants a, b, c with $0<a<b<+\infty$, $0<c<1$ $\left(\textit{in fact } \frac{1}{2}<c<1\right)$ *such that if* Γ *is a measurable subset of* Ω *for which*

$$|\Gamma|_s \equiv \int_\Gamma d\omega \geqq c|\Omega|_s, \tag{4}$$

then for each $\omega\in\Omega$ *there exist* $\omega_1, \omega_2\in\Gamma$ *and* $r_1, r_2\in[a, b]$ *such that*

$$\omega = r_1\omega_1 + r_2\omega_2. \tag{5}$$

Now let $M=\int_{\mathbf{R}^n}\{\varphi^p(x)/F(x)\}\,dx$, so that by (2) and the preceding we have

$$M=\int_\Omega M(\omega)\,d\omega. \tag{6}$$

As in [3], *we first show there is a constant* $D=D(n, p, K)$ such that

$$M(\omega)\leqq DM \textit{ for } \omega\in\Omega. \tag{7}$$

To this end, let $\Gamma=\{\omega\in\Omega: M(\omega)\leqq M[(1-c)|\Omega|_s]^{-1}\}$, with c as in (4), and let $\Gamma'=\Omega\setminus\Gamma$. Then

$$\omega\in\Gamma' \Rightarrow M(\omega)>\frac{M}{(1-c)|\Omega|_s},$$

so by (6),

$$M\geqq\int_{\Gamma'} M(\omega)\,d\omega>\frac{M}{(1-c)|\Omega|_s}\int_{\Gamma'} d\omega=\frac{M(|\Omega|_s-|\Gamma|_s)}{(1-c)|\Omega|_s},$$

whence $|\Gamma|_s>c|\Omega|_s$. By the lemma, given $\omega\in\Omega$ we may write $\omega=r_1\omega_1+r_2\omega_2$ where $\omega_1, \omega_2\in\Gamma$ and $r_1, r_2\in[a, b]$. Hence for $r\geqq 0$, by subadditivity,

$$\varphi(r\omega)=\varphi(rr_1\omega_1+rr_2\omega_2)\leqq\varphi(rr_1\omega_1)+\varphi(rr_2\omega_2),$$

$$\varphi^p(r\omega)\leqq C_p\{\varphi^p(rr_1\omega_1)+\varphi^p(rr_2\omega_2)\},$$

where $C_p=2^{p-1}$ if $p\geqq 1$ or $C_p=1$ if $0<p\leqq 1$. By (1), (2),

$$M(\omega)\leqq C_p\int_0^{+\infty}\frac{\varphi^p(rr_1\omega_1)}{K(r_1)F(rr_1\omega_1)}r^{n-1}\,dr+C_p\int_0^{+\infty}\frac{\varphi^p(rr_2\omega_2)}{K(r_2)F(rr_2\omega_2)}r^{n-1}\,dr.$$

Setting $r=r_i^{-1}s$ this reduces to

$$M(\omega)\leqq C_p\left\{\frac{M(\omega_1)}{r_1^n K(r_1)}+\frac{M(\omega_2)}{r_2^n K(r_2)}\right\}\leqq\frac{MC_p}{(1-c)|\Omega|_s}\left\{\frac{1}{r_1^n K(r_1)}+\frac{1}{r_2^n K(r_2)}\right\}$$

because $\omega_1, \omega_2\in\Gamma$. This proves (7) with

$$D=\frac{C_p}{(1-c)|\Omega|_s}\cdot\frac{2}{\inf\limits_{[a,b]} r^n K(r)}. \tag{8}$$

Recalling the definition of M, and (7), the right-hand inequality of (3) now follows with $B/A=nD$ where D is defined by (8).

To prove the left-hand inequality of (3) let $x \neq 0$ be given, say

$$x = r\,\omega = r \sum_{i=1}^{n} c_i(\omega)\,\omega_i.$$

As above, we have

$$\varphi^p(r\,\omega) \leqq C_p \sum_{i=1}^{n} \varphi^p(rc_i\,\omega_i).$$

For each i, we may write $\omega = c_i\omega_i + |v|\,\tilde{\omega}_i$ where $v = \sum_{j\neq i} c_j\omega_j$ and $\tilde{\omega}_i = |v|^{-1}v$ if $v \neq 0$, or $\omega = c_i\omega_i$ if $v = 0$. Hence by (1),

$$F(r\,\omega) \geqq K(|c_i(\omega)|)\,F(rc_i\,\omega_i), \qquad 1 \leqq i \leqq n,$$

so

$$M(\omega) \leqq C_p \sum_{i=1}^{n} \int_0^{+\infty} \frac{\varphi^p(rc_i\,\omega_i)\,r^{n-1}}{K(|c_i(\omega)|)\,F(rc_i\,\omega_i)}\,\mathrm{d}r.$$

Set $r\,|c_i(\omega)| = s$, and this reduces to

$$M(\omega) \leqq C_p \sum_{i=1}^{n} \left\{ \int_0^{+\infty} \frac{\varphi^p(s(\pm\omega_i))\,s^{n-1}}{F(s(\pm\omega_i))}\,\mathrm{d}s \right\} \frac{1}{|c_i(\omega)|^n\,K(|c_i(\omega)|)}$$

$$\leqq C_p \sum_{i=1}^{n} \frac{\max[M(\omega_i),\; M(-\omega_i)]}{|(c_i(\omega)|^n\,K(|c_i(\omega)|)}, \qquad \omega \in \Omega.$$

Hence, by (6),

$$M \leqq C_p \sum_{i=1}^{n} \max[M(\omega_i),\; M(-\omega_i)] \int_\Omega \frac{\mathrm{d}\,\omega}{|c_i(\omega)|^n\,K(|c_i(\omega)|)},$$

so that the first inequality of (3) will follow, with

(9) $$A = C_p \max_{1\leqq i\leqq n} I_i,$$

provided

(10) $$I_i \equiv \int_\Omega \frac{\mathrm{d}\,\omega}{|c_i(\omega)|^n\,K(|c_i(\omega)|)} < +\infty \quad \text{for} \quad 1 \leqq i \leqq n.$$

Precisely as in [3], by a rotation we can arrange that $c_i(\omega) = \cos\varphi_1$ so that

$$I_i = \int_0^{2\pi}\int_0^{\pi} \cdots \int_0^{\pi} \frac{\sin\varphi_{n-2}\cdots\sin^{n-2}\varphi_1}{|\cos\varphi_1|^n\,K(|\cos\varphi_1|)}\,\mathrm{d}\varphi_1\cdots\mathrm{d}\varphi_{n-1}$$

$$= 2\int_0^{2\pi}\int_0^{\pi} \cdots \int_0^{\pi}\int_0^{\pi/2} \frac{\sin\varphi_{n-2}\cdots\sin^{n-2}\varphi_1}{(\cos\varphi_1)^n\,K(\cos\varphi_1)}\,\mathrm{d}\varphi_1\cdots\mathrm{d}\varphi_{n-1}.$$

The condition $\int_0^1 r^{-n}\,\mathrm{d}r/K(r) < +\infty$ clearly implies that I_i is finite, and the proof is complete.

As a simple example, take $F(x)=b^2|x|^{\beta}+c^2|x|^{\gamma}$, where $b, c\neq 0$ and $\beta>\gamma>n-1$. Condition (1) is easily seen to be satisfied with

$$K(r)=\min(r^{-\beta}, r^{-\gamma})\,(=r^{-\gamma} \quad \text{for} \quad 0<r\leqq 1).$$

We conclude by noting that if we denote $M(\omega)$ in (2) by $M_p(\omega)$, and similarly write

$$M_p=\int\limits_{\mathbf{R}^n}\frac{\varphi^p(x)}{F(x)}\,\mathrm{d}x=\int\limits_{\Omega}M_p(\omega)\,\mathrm{d}\omega,$$

then for $0<p<q<+\infty$, we will have

$$M_q^{1/q}\leqq A(p, q)\,M_p^{1/p} \tag{11}$$

provided the corresponding one-dimensional inequality

$$M_q^{1/q}(\omega)\leqq B(p, q)\,M_p^{1/p}(\omega) \qquad (\omega\in\Omega), \tag{12}$$

holds. Indeed, if we rewrite (7) as $M_p(\omega)\leqq D_p M_p$ and (12) holds, then it is easy to verify that (11) holds with $A(p, q)=B(p, q)\cdot D_p^{(1/p-1/q)}$. In the special case $F(x)=|x|^{n+p\alpha}$ considered by GOSSELIN, it was noted that (12)-hence also (11)-*does* hold, but I do not know whether this is true for those F satisfying (1). A generalization of GOSSELIN's case of inequality (12) was proved in [**1**], but does not include (12) for F satisfying (1).

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