UNIV. BEOGRAD. PUBL. ELEKTROTEHN. FAK.
Ser. Mat. Fiz. № 544 — № 576 (1976), 37–38.

550. SOME REMARKS ON A PAPER OF J. F. RIGBY*

Oene Bottema

1. In a recent paper [**1**] RIGBY derives a set of inequalities for the sides of a triangle which, applied to a special case, gives the following formula for s, R, r:

$$s^2 \leqq (1-\theta^2)^{-1}\big(4R^2+4(1-\theta-4\theta^2)Rr+(3+8\theta+5\theta^2)r^2\big), \tag{1}$$

for all values of θ satisfying $0 \leqq \theta < 1$. For $\theta = 0$ we obtain BLUNDON's inequality

$$s^2 \leqq 4R^2+4Rr+3r^2. \tag{2}$$

Author remarks that the right hand sides of (1) for different values of θ can not be compared and that (2) is „just one of a whole range of best possible inequalities (1)". The term „best possible", as KLAMKIN has emphasized more than once, must be used with much care and has always a relative sense. RIGBY is formally right because it follows from this context that „best possible" is meant by him in relation to a set of inequalities with two parameters θ, ε of which the second does not appear in (1). If we restrict ourselves to (1) proper it must be remarked that there is no „best possible" in it. Indeed, if the right hand side of (1) is denoted by $A(\theta)$ we have

$$A(\theta)-A(0)=4(1-\theta^2)^{-1}\theta(R-2r)\big(\theta R-(1+\theta)r\big), \tag{3}$$

and as $(1+\theta) > 2\theta$ it depends on the ratio $R:r$ whether the difference is more or less than zero.

2. Formula (1) is an inequality (and a rather complicated one) of the type

$$s^2 \leqq \alpha R^2+\beta Rr+\gamma r^2, \tag{4}$$

where the coefficients on the right hand side are functions of a parameter. A much simpler example is the following [**2**]:

$$s^2 \leqq 4R^2+\frac{1}{2}(11-\mu)Rr+\mu r^2, \tag{5}$$

with a linear parameter μ, satisfying $\mu \leqq 3$. For $\mu = 3$ we have again (2). In this case it is the best possible of the range (5); indeed, if $B(\mu)$ denotes the right hand side of (5) we have

$$B(\mu)-B(3)=\frac{1}{3}(3-\mu)(R-2r)r \geqq 0. \tag{6}$$

* Presented May 5, 1976 by D. S. MITRINOVIĆ.

Comparing (1) and (5) we remark that both contain an inequality of the type $s^2 \leqq \alpha R^2 + \gamma r^2$. RIGBY obtains the relation

$$(23 + \sqrt{17})\, s^2 \leqq 128\, R^2 + (109 + 27\sqrt{17})\, r^2, \tag{7}$$

which we reduce to

$$s^2 \leqq \frac{1}{4}(23 - \sqrt{17})\, R^2 + (4 + \sqrt{17})\, r^2. \tag{8}$$

From (5), for $\mu = 11$, we obtain

$$s^2 \leqq 4\, R^2 + 11\, r^2, \tag{9}$$

which is not only much simpler than (8), but also better.

Indeed, the difference between the right hand sides of (8) and (9) is seen to be

$$\frac{1}{4}(7 - \sqrt{17})(R^2 - 4\, r^2), \tag{10}$$

which is positive for a non-equilateral triangle.

3. RIGBY gives also minima for s^2 in terms of R and r and derives

$$s^2 \geqq \alpha_1 R^2 + \beta_1 Rr + \gamma_1 r^2, \tag{11}$$

the coefficients being quadratic functions of a parameter.

We remark that a much simpler set reads [**2**]

$$s^2 \geqq \frac{1}{2}(27 - \lambda)\, Rr + \lambda\, r^2, \quad \lambda \geqq -5. \tag{12}$$

From both (11) and (12) follows the STEINIG—BLUNDON inequality

$$s^2 \geqq 16\, Rr - 5\, r^2. \tag{13}$$

But while RIGBY derives from (11) that $s^2 \geqq 27\, r^2$, it follows from (12), for $\lambda = 0$, the stronger inequality $s^2 \geqq \frac{27}{2} Rr$.

The conclusion of our remarks may be that RIGBY's work on sextic inequalities, although of interest in it self, does not provide us with an optimal procedure to obtain (s, R, r) inequalities of the type considered by him.

REFERENCES

1. J. F. RIGBY: *Sextic inequalities for the sides of a triangle.* These Publications № **498**—№ **541** (1975), 51—58.

2. O. BOTTEMA: *Inequalities for R, r and s.* These Publications № **338**—№ **352** (1971), 27—36.