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544. COMPLÉMENTS AU TRAITÉ DE KAMKE, XIV. APPLICATIONS OF THE VARIATION OF PARAMETERS METHOD TO NONLINEAR SECOND ORDER DIFFERENTIAL EQUATIONS*

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1. In solving the differential equation

$$y'' + 3\,y' + 2\,y = 2\,y^3,$$

proposed by V. JAMET [1], P. MANSION [2] applied an interesting variant of the classical variation of parameters method. In the same note [2] he also solved a more general equation

$$y'' + (k+l)\,y' + kly = Ay^m, \tag{1.1}$$

where k, l, m and A are constants, provided that one of the conditions $(m+1)\,k = 2\,l$ or $(m+1)\,l = 2\,k$ is fulfilled.

We note that a special case of equation (1.1) is present in the famous paper [3] of P. PAINLEVÉ. Namely, PAINLEVÉ solved the equation

$$y'' + 3\,ay' + 2\,a^2\,y = 2\,y^3 \qquad (a = \text{const}) \tag{1.2}$$

which is obtained from (1.1) for $k = 2\,a$, $l = a$, $A = 2$, $m = 3$.

Equation (1.2) is also recorded in KAMKE's collection [4] as equation 6.24.

In this note we shall extend MANSION's method to some other nonlinear second order differential equations and shall also give some remarks regarding it.

2. Consider the differential equation

$$y'' + A\,(x)\,y' = F\,(x,\,y), \tag{2.1}$$

where A and F are given functions.

Equation (2.1) can be replaced by the system

$$y' + f(x)\,y = z, \quad z' + g\,(x)\,z = F\,(x,\,y), \tag{2.2}$$

where

$$f(x) + g\,(x) = A\,(x), \quad f'\,(x) + f(x)\,g\,(x) = 0. \tag{2.3}$$

REMARK. Functions f and g can always be determined from (2.3), since that system can easily be reduced to BERNOULLI's equation $f'(x) + A\,(x)\,f(x) - f(x)^2 = 0$.

In order to solve the system (2.2) we apply MANSION's method. We first solve the "linear parts" of that system, i.e. we solve the system

$$y' + f(x)\,y = 0, \quad z' + g\,(x)\,z = 0,$$

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to obtain

$$y=Ce^{-\int f(x)\,dx}, \quad z=De^{-\int g(x)\,dx}, \tag{2.4}$$

where C and D are arbitrary constants. Suppose that C and D are differentiable functions of x. Then from (2.4) and (2.2) follows

$$C'=De^{\int (f(x)-g(x))\,dx}, \quad D'=F(x, Ce^{-\int f(x)\,dx})\, e^{\int g(x)\,dx}, \tag{2.5}$$

and hence

$$\frac{dC}{dD}=\frac{De^{\int (f(x)-2g(x))\,dx}}{F(x, Ce^{-\int f(x)\,dx})}. \tag{2.6}$$

Therefore, if the expression

$$\frac{e^{\int (f(x)-2g(x))\,dx}}{F(x, Ce^{-\int f(x)\,dx})}$$

does not depend on x, but only on C, then in certain cases equation (2.6) can be integrated. If (2.6) yields an explicit connection between C and D, then (2.5) can be used to determine C. Then from (2.4) we get the general solution of equation (2.1).

EXAMPLE 1. Consider the differential equation

$$y''+\frac{3}{x}y'=\sum_k a_k(x)y^k.$$

It can be replaced by the system

$$y'+\frac{2}{x}y=z, \quad z'+\frac{1}{x}z=\sum_k a_k(x)y^k.$$

Using the above method we find

$$\frac{dC}{dD}=\frac{D}{\sum_k \frac{a_k(x)}{x^{2k}}C^k}. \tag{2.7}$$

Hence, if $a_k(x)=a_k x^{2k}$ ($a_k=\text{const}$), equation (2.7) becomes

$$\frac{dC}{dD}=\frac{D}{\sum_k a_k C^k},$$

and can be integrated by quadratures.

For instance, for the equation

$$y''+\frac{3}{x}y'=ax^6y^3 \qquad (a=\text{const}) \tag{2.8}$$

we find

$$\frac{dC}{dD}=\frac{D}{aC^3},$$

which implies $D=\sqrt{\frac{1}{2}aC^4+k_1}$, where k_1 is an arbitrary constant.

From (2.5) we get

$$C' = \sqrt{\frac{1}{2} aC^4 + k_1}\, x,$$

i.e.

(2.9)
$$\int \frac{1}{\sqrt{\frac{1}{2} aC^4 + k_1}} \, \mathrm{d}C = \frac{1}{2} x^2 + k_2,$$

(k_1, k_2 arbitrary constants). Hence, the general solution of equation (2.8) is $y = x^{-2} C$, where C is defined by (2.9).

3. We may start from the other end. Namely, we first solve a system of the form

(3.1)
$$y' + f(x)\, y = F(x, y, z), \quad z' + g(x)\, z = G(x, y, y', z)$$

using MANSION's method. Certain conditions on F and G will be imposed on the way. Then, in case the system (3.1) is solved successfully, we eliminate from it the function z, and obtain a second order differential equation in y, which can be solved by quadratures.

EXAMPLE 2. Consider the system

(3.2)
$$y' + \frac{1}{x} y = yz, \quad z' - \frac{1}{x} z = G(x)\, y^2.$$

Since the solution of the system

$$y' + \frac{1}{x} y = 0, \quad z' - \frac{1}{x} z = 0$$

is given by

(3.3)
$$y = \frac{C}{x}, \quad z = Dx,$$

where C and D are arbitrary constants, suppose that the functions y and z, defined by (3.3), where C and D are differentiable functions of x, satisfy (3.2). We find

(3.4)
$$C' = CDx, \quad D' = x^{-3} G(x)\, C^2,$$

and hence

(3.5)
$$\frac{\mathrm{d}C}{\mathrm{d}D} = \frac{Dx^4}{G(x)\, C}.$$

We therefore take that $G(x) = ax^4$ ($a = \text{const}$). Equation (3.5) becomes

$$\frac{\mathrm{d}C}{\mathrm{d}D} = \frac{D}{aC},$$

and we get

(3.6)
$$D = \sqrt{aC^2 + k_1} \qquad (k_1 \text{ arbitrary constant}).$$

Substituting (3.6) into the first equation of (3.4), we get

$$C' = C\sqrt{aC^2 + k_1}\, x,$$

i.e.

(3.7)
$$\int \frac{1}{C\sqrt{aC^2 + k_1}} \, \mathrm{d}C = \frac{1}{2} x^2 + k_2 \qquad (k_1,\ k_2 \text{ arbitrary constants}).$$

On the other hand, elimination of z from the system (3.3) yields the equation

(3.8) $$x^2 yy''-x^2 (y')^2-xyy'-2y^2=ax^6 y^4.$$

Therefore, the general solution of equation (3.8) is $y=\frac{C}{x}$, where C is defined by (3.7).

Example 3. If we apply the above method to the system

(3.9) $$y'-\frac{1}{x}y=F(x)\,y^p z,\quad z'-\frac{1}{x}z=G(x)\,y^q \qquad (p,\,q=\text{const}),$$

we obtain the condition

$$G(x)=aF(x)\,x^{p-q+1} \qquad (a=\text{const}).$$

On the other hand, system (3.9) leads to the equation

(3.10) $$x^2 F(x)\,yy''-px^2 F(x)\,(y')^2+\big((p-2)\,xF(x)-x^2 F'(x)\big)\,yy'+\big(2F(x)+xF'(x)\big)\,y^2 = ax^{p-q+3} F(x)^3 y^{p+q+1},$$

whose general solution is $y=Cx$, where C is defined by

$$\int C^{-p}\Big(\frac{2aC^{q-p+1}}{q-p+1}+k_1\Big)^{-1/2} dC=\int x^p F(x)\,dx+k_2 \qquad (k_1,\,k_2 \text{ arbitrary constants}).$$

4. We give an other application of the exposed method. Namely, the differential equation which connects C and D has a role of an invariant, and can be used to identify equivalent equations. For instance, in the previous example (Example 3) the equation for C and D, under the condition $G(x)=aF(x)\,x^{p-q+1}$, reads

(4.1) $$\frac{dC}{dD}=\frac{1}{a}C^{p-q}D.$$

This means that all differential equations of the form (3.10) where $F(x)\not\equiv 0$ is arbitrary and $p-q$ takes the same value, are mutually equivalent, because they can all be reduced to the same equation (4.1). In particular, instead of the equation (3.10) we can consider the simpler equation

$$x^2 yy''-px^2 (y')^2+(p-2)\,xyy'+2y^2=ax^{p-q+3}y^{p+q+1},$$

which is obtained from (3.10) for $F(x)\equiv 1$.

We give one more example.

Example 4. Differential equation

(4.2) $$y''+yy'-y^3=0$$

is a special case of the equation

(4.3) $$y''+\big(y+3f(x)\big)\,y'-y^3+f(x)\,y^2+\big(f'(x)+2f(x)^2\big)\,y=0,$$

as it is obtained from it for $f(x)\equiv 0$.

However, equations (4.2) and (4.3) are equivalent. Indeed, if equation (4.3) is replaced by the system

$$y'+f(x)\,y=z,\quad z'+2f(x)\,z=y^3-f(x)\,y^2-yy'$$

to which we apply the variation of parameters method, we arrive at the equation

(4.4) $$\frac{dC}{dD}=\frac{D}{C^3-CD},$$

which does not contain the function f, meaning that the equation (4.3) can be reduced to (4.4) for every function f, and hence for $f(x)\equiv 0$.

In KAMKE's collection [4] equations (4.2) and (4.3) are recorded as equations 6.30 and 6.33, respectively. It is also noted that equation (4.3) can be reduced to (4.2) by the substitution $y=u'(x)z$, where z is the new unknown function, and u satisfies the equation $u''+f(x)u'=0$.

REMARK. Note that the equation (4.2) by an application of the standard substitution $y'=p$, $y''=p\dfrac{\mathrm{d}p}{\mathrm{d}y}$ is reduced to

$$\frac{\mathrm{d}y}{\mathrm{d}p}=\frac{p}{y^3-yp},$$

and that is precisely equation (4.4).

REFERENCES

1. V. JAMET: *Question* 364. Nouvelle Correspondance mathématique **6** (1880), 381—382.
2. P. MANSION: *Une lettre de Mansion.* Nouvelle Correspondance mathématique **6** (1880), 457—458.
3. P. PAINLEVÉ: *Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme.* Acta Math. **25** (1902), 1—85.
4. E. KAMKE: *Differentialgleichungen—Lösungsmethoden und Lösungen.* Leipzig 1959.